Session 18
Nov. 7, 2019

Last time we discussed Itô's lemma for an Itô process $dX = f\,dt + g\,dW$.

It reads: $\boxed{dF = \left[\frac{\partial F}{\partial t} + f\frac{\partial F}{\partial x} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} g^2\right] dt + g\frac{\partial F}{\partial x}\,dW}$

for $f=0$, $g=1$, we get $dF = \left[\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\right]dt + \frac{\partial F}{\partial x}\,dW$

<u>Ex.:</u> geometric Brownian motion $S(W(t),t) = e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$

$\Rightarrow$ corresponding SDE is $dS = \left[(\mu - \frac{\sigma^2}{2})S + \frac{1}{2}\sigma^2 S\right]dt + \sigma S\,dW$

$$= \mu S\,dt + \sigma S\,dW$$

What is $\mathbb{E}(S(t)^n)$?

Write $F(S(t),t) = S(t)^n$

$$dS^n = \Big[\underbrace{\mu S}_{f}\underbrace{n S^{n-1}}_{F'} + \frac{1}{2}\underbrace{\sigma^2 S^2}_{g^2}\underbrace{n(n-1)S^{n-2}}_{F''}\Big]dt + \underbrace{\sigma S}_{g}\underbrace{n S^{n-1}}_{F'}dW$$

$$= S^n\left(\mu n + \frac{1}{2}\sigma^2 n(n-1)\right)dt + n\sigma S^n\,dW$$

$$= S^n\left[\left(\mu n + \frac{1}{2}\sigma^2 n(n-1)\right)dt + n\sigma\,dW\right]$$

$\Rightarrow$ GBM with different parameters

$$\Rightarrow \mathbb{E}(S(t)^n) - \mathbb{E}(S_0^n)$$

$$= \left(n\mu + \tfrac{1}{2}\sigma^2 n(n-1)\right) \int_0^t \mathbb{E}(S(t)^n)\,dt + n\sigma \int_0^t \mathbb{E}(S(t)^n)\underbrace{\mathbb{E}(dW(s))}_{=0}$$

$$\Rightarrow \mathbb{E}(S(t)^n) = S_0^n e^{(n\mu + \frac{1}{2}n(n-1)\sigma^2)t}$$

in particular: $\cdot\ \mathbb{E}(S(t)) = S_0 e^{\mu t} \left(\approx S_0 + S_0\mu t \text{ for small } t\right)$

$$\cdot\ \mathrm{Var}(S(t)) = \mathbb{E}(S(t)^2) - \mathbb{E}(S(t))^2$$

$$= S_0^2 e^{(2\mu+\sigma^2)t} - S_0^2 e^{2\mu t}$$

$$= S_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right) \quad \left(\approx (S_0\sigma)^2 t \text{ for small } t\right)$$