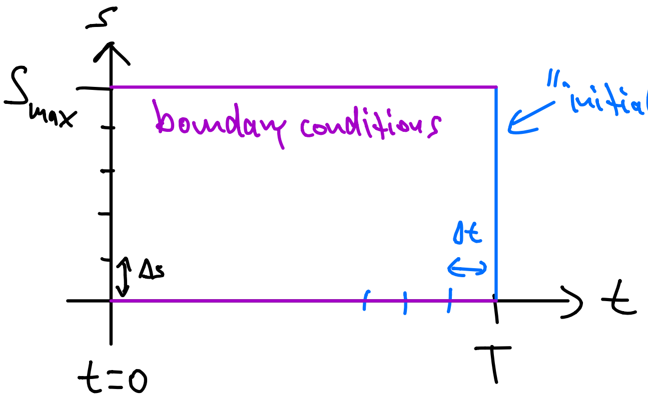
Session 20
Nov. 18, 2019

4.3 Discrete Finite Differences



"initial" condition $C(s,T) = \text{payoff}$

boundary conditions:

$C(t,0) = 0$

boundary condition at S_{max} is only due to discretization; good choices are:

- $C(t, S_{max}) = S_{max} - Ke^{-r(T-t)}$, this is a sol. to BS eq.
- $C(t, S_{max}) = S_{max}$, simpler; justified since S_{max} should be chosen $\gg K$

Discretization:

- partition $[0,T]$ into M steps of size $\Delta t = \frac{T}{M}$, $t_j = j\Delta t$
- partition $[0, S_{max}]$ into N steps of size $\Delta s = \frac{S_{max}}{N}$, $s_i = i\Delta s$
- we abbreviate $C(t_j, s_i) = C_i^j$

Then:

$$\frac{\partial C_i^j}{\partial t} = \frac{C_i^{j+1} - C_i^j}{\Delta t} + \mathcal{O}(\Delta t) \qquad \text{for fixed } i$$

$$\frac{\partial C_i^j}{\partial s} = \frac{C_{i+1}^j - C_i^j}{\Delta s} + \mathcal{O}(\Delta s) \qquad \text{for fixed } j$$

the spatial derivative can be improved (fix j here):

$$(1)\quad C(s_i+\Delta s) \overset{\text{Taylor}}{=} C(s_i) + \frac{\partial c}{\partial s}(s_i)\Delta s + \frac{1}{2}\frac{\partial^2 c}{\partial s^2}(s_i)\Delta s^2 + \frac{1}{3!}\frac{\partial^3 c}{\partial s^3}\Delta s^3 + O(\Delta s^4)$$

$$(2)\quad C(s_i-\Delta s) = C(s_i) - \frac{\partial c}{\partial s}(s_i)\Delta s + \frac{1}{2}\frac{\partial^2 c}{\partial s^2}(s_i)\Delta s^2 - \frac{1}{3!}\frac{\partial^3 c}{\partial s^3}\Delta s^3 + O(\Delta s^4)$$

$$(1)-(2) \Rightarrow C(s_i+\Delta s) - C(s_i-\Delta s) = 2\frac{\partial c}{\partial s}(s_i)\Delta s + O(\Delta s^3)$$

the centralized derivative $\frac{\partial c_i^j}{\partial s} = \frac{c_{i+1}^j - c_{i-1}^j}{2\Delta s} + O(\Delta s^2)$ improves the error

second derivative: $(1)+(2)$

$$\Rightarrow C(s_i+\Delta s) + C(s_i-\Delta s) = 2C(s_i) + \frac{\partial^2 c}{\partial s^2}(s_i)\Delta s^2 + O(\Delta s^4)$$

$$\Rightarrow \frac{\partial^2 c_i^j}{\partial s^2} = \frac{c_{i+1}^j - 2c_i^j + c_{i-1}^j}{\Delta s^2} + O(\Delta s^2)$$

↳ also $O(\Delta s^2)$ error, as for centralized first derivative

4.4 Stability of Time-stepping Methods

Stability = convergence to true solution

We just consider the simple example of exponential decay

$$\frac{dy}{dt} = -\lambda y, \quad \lambda > 0 \Rightarrow \text{solution: } y(t) = y_0 e^{-\lambda t}$$

we consider also $\lambda >> 1$

there are two ways to solve this ODE:

- Explicit Euler method: $\frac{Y^{j+1} - Y^j}{\Delta t} = -\lambda Y^j$ (r.h.s. evaluated at j)

$\Rightarrow Y^{j+1} = -\lambda Y^j \Delta t + Y^j = (1 - \lambda \Delta t) Y^j$

$\Rightarrow Y^M = (1 - \lambda \Delta t)^M Y_0$ (note: $\lim_{M \to \infty} Y^M = \lim_{M \to \infty} (1 - \lambda \frac{t}{M})^M Y_0 = e^{-\lambda t} Y_0$)

We know that $Y^M \to 0$ for large T

This gives us a condition for convergence, i.e., stability: $|1 - \lambda \Delta t| < 1$

↳ need $1 - \lambda \Delta t < 1$, i.e., $\lambda > 0$ ✓

↳ need $-1 + \lambda \Delta t < 1$, i.e., $\Delta t < \frac{2}{\lambda}$

$\Rightarrow$ only for small enough Δt is the discretization stable

- Implicit Euler method: $\frac{Y^{j+1} - Y^j}{\Delta t} = -\lambda Y^{j+1}$ (r.h.s evaluated at $j+1$)

$\Rightarrow (1 + \lambda \Delta t) Y^{j+1} = Y^j \Rightarrow Y^{j+1} = \left(\frac{1}{1 + \lambda \Delta t}\right) Y^j$

$\Rightarrow Y^M = \left(\frac{1}{1 + \lambda \Delta t}\right)^M Y^0$

now stability condition is $\left|\frac{1}{1 + \lambda \Delta t}\right| < 1$, which always holds here, since $\lambda > 0$

$\Rightarrow$ implicit scheme is unconditionally stable

4.5 Application to Heat Equation

consider $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$, initial value $V(x,0)$ (want to know $V(x,T)$) / (for BS: backwards)

take boundary conditions at $x_0 = 0$ and $x_{max} = x_{n+1}$

we have:
$$\frac{\partial V}{\partial t}(x_i, t_j) = \frac{V(x_i, t_{j+1}) - V(x_i, t_j)}{\Delta t} + O(\Delta t)$$

$$\frac{\partial^2 V}{\partial x^2}(x_i, t) = \frac{V(x_{i+1}, t) - 2V(x_i, t) + V(x_{i-1}, t)}{\Delta x^2} + O(\Delta x^2)$$

↳ $t = t_j$: explicit scheme
↳ $t = t_{j+1}$: implicit scheme

denote again $V(x_i, t_j) = V_i^j$

explicit:
$$\frac{V_i^{j+1} - V_i^j}{\Delta t} = \frac{V_{i+1}^j - 2V_i^j + V_{i-1}^j}{\Delta x^2}$$

$$\Rightarrow \quad V_i^{j+1} = \frac{\Delta t}{\Delta x^2} V_{i+1}^j + \left(1 - \frac{2\Delta t}{\Delta x^2}\right) V_i^j + \frac{\Delta t}{\Delta x^2} V_{i-1}^j$$

we will see numerically that $\frac{\Delta t}{\Delta x^2} < \text{const}$ is needed for stability

implicit:
$$\frac{V_i^{j+1} - V_i^j}{\Delta t} = \frac{V_{i+1}^{j+1} - 2V_i^{j+1} + V_{i-1}^{j+1}}{\Delta x^2}$$

$$\Rightarrow \quad V_i^j = \underbrace{-\frac{\Delta t}{\Delta x^2}}_{a} V_{i+1}^{j+1} + \left(1 + \frac{2\Delta t}{\Delta x^2}\right) V_i^{j+1} - \frac{\Delta t}{\Delta x^2} V_{i-1}^{j+1}$$

In matrix notation:

here boundary eq.s are still wrong

$$\underbrace{\begin{pmatrix} (1+2a) & -a & & & 0 \\ -a & (1+2a) & -a & & \\ & -a & \ddots & \ddots & \\ & & \ddots & \ddots & -a \\ 0 & & & -a & (1+2a) \end{pmatrix}}_{n\times n \text{ matrix } A} \underbrace{\begin{pmatrix} V_1^{j+1} \\ V_2^{j+1} \\ \vdots \\ \vdots \\ V_n^{j+1} \end{pmatrix}}_{\text{vector } \vec{V}^{j+1}} = \underbrace{\begin{pmatrix} V_1^{j} \\ V_2^{j} \\ \vdots \\ \vdots \\ V_n^{j} \end{pmatrix}}_{\text{vector } \vec{V}^{j}}$$

$\Rightarrow$ need to solve tridiagonal system of equations to get $\vec{V}^{j+1}$ from $\vec{V}^j$

what happens at the boundary?

V_0^{j+1} and V_{n+1}^{j+1} are given by fixed boundary conditions!

we have $V_1^j = -a V_2^{j+1} + (1+2a) V_1^{j+1} - a V_0^{j+1}$

$$V_n^j = -a V_{n+1}^{j+1} + (1+2a) V_n^{j+1} - a V_{n-1}^{j+1}$$

so with boundary conditions the tridiagonal system is

$$\begin{pmatrix} \ddots & \ddots & & \\ \ddots & (1+2a) & -a & \\ & -a & \ddots & \ddots \\ & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} V_1^{j+1} \\ \vdots \\ \vdots \\ V_n^{j+1} \end{pmatrix} = \begin{pmatrix} V_1^j + a \underbrace{V_0^{j+1}}_{\text{fixed}} \\ V_2^j \\ \vdots \\ V_{n-1}^j \\ V_n + a \underbrace{V_{n+1}^{j+1}}_{\text{fixed}} \end{pmatrix}$$

$$\Rightarrow \boxed{A \, \vec{V}^{j+1} = \vec{V}^{j} + \begin{pmatrix} \alpha V_0^{j+1} \\ 0 \\ \vdots \\ 0 \\ \alpha V_{n+1}^{j+1} \end{pmatrix}}$$

tridiagonal

↳ stable scheme with right boundary conditions