5. Parameter Estimates for Time Series

Session 21
Nov. 25, 2019

stock price model: geom. BM $\quad dS = \mu S\, dt + \sigma S\, dW$

$$S(t) = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$$

Time Series: we sample $S(t)$ at times $t_1, \dots, t_n$ which gives us $S(t_i) = S_i$

Then let us consider the log-returns r_i s.t. $S(t_i) = S(t_{i-1}) e^{r_i}$

$$\Rightarrow r_i = \ln \frac{S(t_i)}{S(t_{i-1})} = \ln S_i - \ln S_{i-1}$$

for GBM this is $r_i = \ln S_0 e^{(\mu - \frac{\sigma^2}{2})t_i + \sigma dW(t_i)} - \ln S_0 e^{(\mu - \frac{\sigma^2}{2})t_{i-1} + \sigma dW(t_{i-1})}$

$$= \left(\mu - \frac{\sigma^2}{2}\right)(t_i - t_{i-1}) + \sigma\left(dW(t_i) - dW(t_{i-1})\right)$$

$$= \left(\mu - \frac{\sigma^2}{2}\right)\Delta t_i + \sigma \Delta W_i$$

$\Rightarrow$ r_i's are normally and independently distributed

Let us choose $\Delta t_i = \Delta t$. Then the theoretical prediction is:

- Expectation $\mathbb{E}(r_i) = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma \underbrace{\mathbb{E}(\Delta W_i)}_{=0} = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t$

- Variance $\mathrm{Var}(r_i) = \sigma^2 \underbrace{\mathrm{Var}(\Delta W_i)}_{\Delta t} = \sigma^2 \cdot \Delta t$

From our data we get:

- sample mean $\bar{r} = \frac{1}{n} \sum_{i=1}^{n} r_i$

- sample variance $\sigma_r^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (\bar{r} - r_i)^2$

$\frac{1}{n-1}$ prefactor better when considering samples ("unbiased sample variance")

For large n we expect $\mathbb{E}(r_i) \approx \bar{r}$ and $Var(r_i) \approx \sigma_r^2$

Therefore we approximate our parameters

- $\sigma = \sqrt{\frac{Var(r_i)}{\Delta t}}$ by $\boxed{\hat{\sigma} = \sqrt{\frac{\sigma_r^2}{\Delta t}} = \frac{\sigma_r}{\sqrt{\Delta t}}}$
- $\mu = \frac{\mathbb{E}(r_i)}{\Delta t} + \frac{\sigma^2}{2}$ by $\boxed{\hat{\mu} = \frac{\bar{r}}{\Delta t} + \frac{\hat{\sigma}^2}{2}}$

note (see HW): $\cdot$ one can show $Var[\hat{\sigma}] = \frac{\mathbb{E}(\hat{\sigma})^2}{2n}$

$\cdot$ but $Var[\hat{\mu}]$ not necessarily smaller the larger n

according to our model the r_i's are normally and independently distributed

↳ need to check if this holds for our data

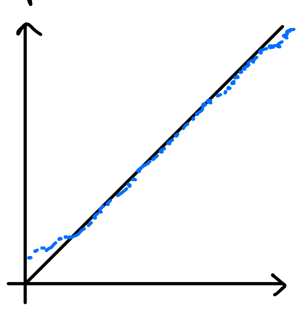
<u>test assumption of normality:</u>

QQ plot (HW11 e))

recall: $\cdot$ rescale $\tilde{r}_i = \frac{r_i - \bar{r}}{\sigma_r}$

$\cdot$ sort $\tilde{r}_i$

$\cdot$ plot vs. sorted sample of standard normal distribution



test assumption of independence:

covariance $\mathrm{Cov}(X,Y) = \mathbb{E}\left[(X-\mathbb{E}(X))(Y-\mathbb{E}(Y))\right]$

$$= \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y)$$

if X, Y are independent, then $\mathbb{E}(XY) = \mathbb{E}(X)\,\mathbb{E}(Y)$ and $\mathrm{Cov}(X,Y) = 0$.

note: $\mathrm{Var}(X) = \mathrm{Cov}(X,X)$

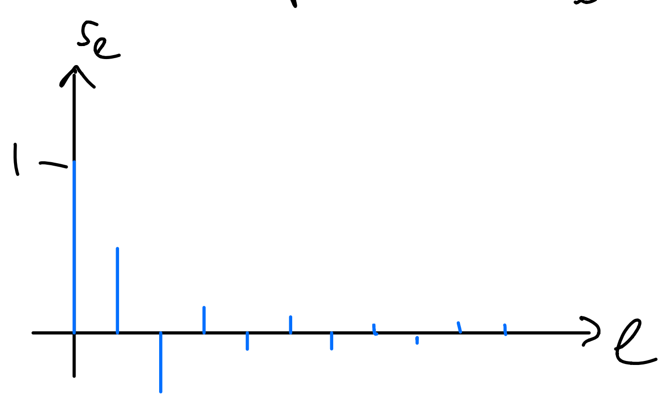
we use autocorrelation fct. (ACF):

normalized, s.t. $\rho_0 = 1$

$$\rho_\ell = \frac{\mathrm{Cov}(r_i, r_{i-\ell})}{\sqrt{\mathrm{Var}(r_i)\,\mathrm{Var}(r_{i-\ell})}}$$

, ℓ is called "lag"

- perfect correlation means $\rho_\ell = 1$ (anticorrelation: $\rho_\ell = -1$)
- more or less independent if $|\rho_\ell| \ll 1$



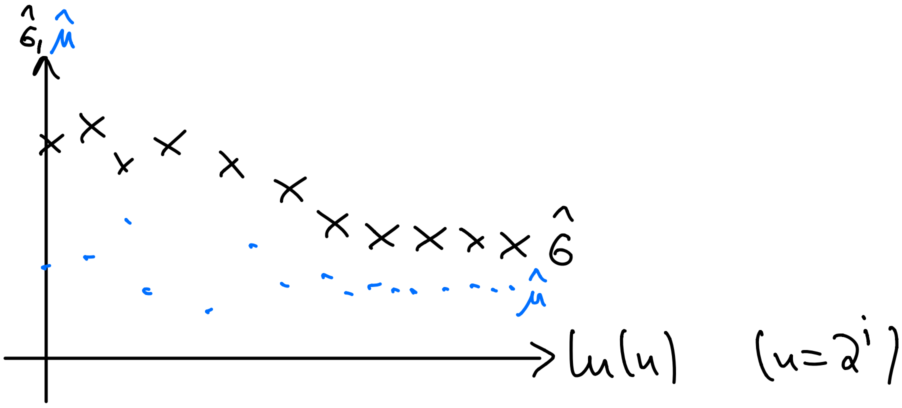
for stocks there can be "inertia" effects, i.e., autocorrelation between nearby r_i's if Δt was chosen too small $\rightarrow$ increase Δt to get more reliable estimate $\hat{\sigma}$

python: acorr[r, maxlags=...]

Homework:

a) one realization of GBM, size 2^k

then estimate $\hat{\mu}, \hat{\sigma}$ for every 2^i-th sample point, $i = 0, \dots, k-1$



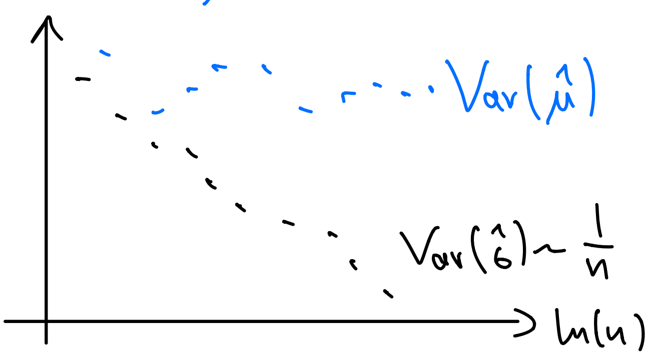
semilogx

b) ensemble of GBMs with some parameters

↳ $Var(\hat{\sigma})$, $Var(\hat{\mu})$

↑ ensemble variance

$\ln Var(\hat{\sigma})$, $\ln Var(\hat{\mu})$

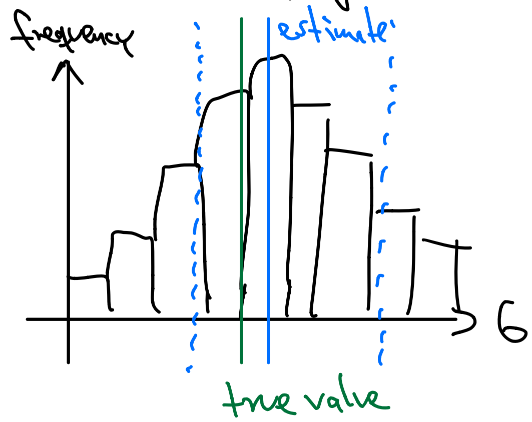


c) "Backtracking"

- given a single time series from part a) ⟶ compute $\hat{\mu}, \hat{\sigma}$
- generate ensemble of GBMs with these parameters
- compute $Var(\hat{\mu})$, $Var(\hat{\sigma})$

=> test how reliable estimate was

python: hist(sigma_distribution, number of bins, histtype='stepfilled')



very thin for σ
but wide for μ

d), e), f) consider some noise sources:

- periodic noise: $S_{per} = \overset{\text{GBM}}{S} + c_1\sqrt{\Delta t}\,\sin(2\pi \overset{\text{frequency}}{f}\,\text{arange}(N+1))$
- Gaussian noise: $S_{Gauss} = S + c_1\sqrt{\Delta t}\,\text{normal}(0,1,N+1)$

- how does the noise change estimates for $\hat{\mu}, \hat{\sigma}$?
- normality?
- independence?