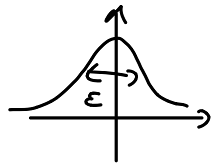
Notes on nonlinear SPDEs:

Φ^4 theory: $\underbrace{\partial_t \Phi = \Delta\Phi}_{\text{heat eq.}} - \Phi^3 + \xi$ ← white noise , SPDE

$\hookrightarrow$ non-linearity

regularization by convolution $\xi_\varepsilon = \xi * \rho_\varepsilon$, $\int \rho_\varepsilon = \int \rho = 1$



$\Rightarrow \partial_t \Phi_\varepsilon = \Delta \Phi_\varepsilon - \Phi_\varepsilon^3 + \xi_\varepsilon$ can be solved as classical PDE (Φ_ε differentiable)

but $\Phi_\varepsilon \to 0$ as $\varepsilon \to 0$ (can be shown in dim. $d = 2, 3$)

introducing a counterterm $C_\varepsilon \Phi_\varepsilon$ in equation can make new $\Phi_\varepsilon \xrightarrow{\varepsilon \to 0} \Phi$

$\hookrightarrow$ idea of renormalization of an SPDE

rigorous solution theory: Martin Hairer (Fields medal 2014)

Notes on the connection from SDE to PDE:

this goes via the distribution fct.:

Ito SDE: $dX = f(X)\,dt + g(X)\,dW$

$$\mathbb{E}(F(X_t)) = \int_{-\infty}^{\infty} \rho(x,t)\, F(x)\,dx$$

$$\Longrightarrow \frac{d\mathbb{E}(F(X_t))}{dt} = \int_{-\infty}^{\infty} \frac{\partial \rho(x,t)}{\partial t} F(x)\,dx$$

with Itô's lemma:
$$\mathbb{E}(dF(X_t)) = \mathbb{E}\left(\frac{dF}{dx}\left(f(x)\,dt + g(x)\,dW\right) + \frac{1}{2}\frac{d^2F}{dx^2} g(x)^2\,dt \right)$$
$$= \int_{-\infty}^{\infty} \rho(x,t)\left(\frac{dF}{dx} f(x) + \frac{1}{2}\frac{d^2F}{dx^2} g(x)^2 \right) dx\,dt$$

integration by parts
$$= \int_{-\infty}^{\infty} \left[-\frac{\partial}{\partial x}\left(\rho(x,t) f(x)\right) + \frac{\partial^2}{\partial x^2}\left(\rho(x,t)\frac{1}{2} g(x)^2\right)\right] F(x)\,dx\,dt$$

comparing the expressions yields

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(g^2 \rho\right) - \frac{\partial}{\partial x}\left(f\rho\right)$$, Fokker-Planck equation

Ex.: Ornstein-Uhlenbeck process: $dX = -\alpha V'(x)\,dt + \sqrt{2\beta}\,dW$

$$\Longrightarrow \frac{\partial \rho}{\partial t} = \beta \frac{\partial^2 \rho}{\partial x^2} + \alpha \frac{\partial}{\partial x}\left(V' \rho\right)$$

if there is equilibrium for large t, then $\frac{\partial \rho}{\partial t} = 0$ (large t)

$$\Rightarrow \frac{\partial}{\partial x}\left(\beta \frac{\partial \rho_e}{\partial x} + \alpha V' \rho_e\right) = 0$$

suppose $\frac{\partial \rho_e}{\partial x}, V', \rho_e \xrightarrow{x \to \pm\infty} 0$

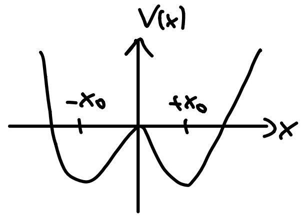
$\Rightarrow$ need to solve $\beta \frac{\partial \rho_e}{\partial x} + \alpha V' \rho_e = 0$

$$\Rightarrow \frac{\partial \rho_e}{\partial x} = -\frac{\alpha}{\beta} V' \rho_e$$

equilibrium

$$\Rightarrow \frac{d\rho_e}{\rho_e} = -\frac{\alpha}{\beta} V' dx \Rightarrow \rho_e(x) = C e^{-\frac{\alpha}{\beta} V(x)}$$

e.g., for $V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$:

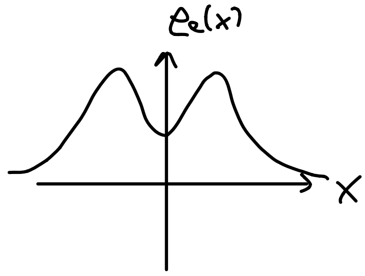


two equilibria

without noise ($g = 0$), the solution would end up either in $-x_0$ or in $+x_0$

with noise: switching between equilibria

↳ distribution fct.



e.g., useful for detecting weak signals in noise