Stochastic Methods Lab, Jacobs University Bremen, Fall 2020

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Annuities:

- ordinary annuity (pays C at end of year):

$$FV = C \sum_{i=0}^{n-1} (1+r)^i = C \, \frac{(1+r)^n - 1}{r}$$

geometric series

- annuity due (pays at beginning of year):

$$FV = C \sum_{i=1}^{n} (1+r)^i = (1+r)\, C \sum_{i=0}^{n-1} (1+r)^i = C(1+r)\, \frac{(1+r)^n - 1}{r}$$

- general annuity: m payments per year:

$$FV = C \sum_{i=0}^{n\cdot m-1} \left(1+\frac{r}{m}\right)^i = C \left(\frac{\left(1+\frac{r}{m}\right)^{n\cdot m} - 1}{\frac{r}{m}} \right) \quad \text{(ordinary)}$$

What is PV?

ordinary annuity: $$PV = \sum_{i=1}^{nm} C \left(1+\frac{r}{m}\right)^{-i}$$

$$= C \sum_{i=1}^{nm} \left(\frac{1}{1+\frac{r}{m}}\right)^i$$

$$= C\left(\frac{1}{1+\frac{r}{m}}\right) \sum_{i=0}^{nm-1} \left(\frac{1}{1+\frac{r}{m}}\right)^i$$

$$= C\left(\frac{1}{1+\frac{r}{m}}\right)\left(\frac{(1+\frac{r}{m})^{-nm}-1}{\frac{1}{1+\frac{r}{m}}-1}\right)$$

$$= C\,\frac{m}{r}\left(1-\left(1+\frac{r}{m}\right)^{-nm}\right)$$

– perpetual annuity: $n\to\infty$

$$PV = \lim_{n\to\infty} C\,\frac{m}{r}\left(1-\left(1+\frac{r}{m}\right)^{-nm}\right) = C\,\frac{m}{r}$$

Amortization:

repay loan with regular payments (e.g., loans for houses)

↳ payments for principal (repay loan) + interest

traditional mortgage = equal regular payments

↳ amount $C = \underbrace{PV}_{\text{loan}}\,\frac{r}{m}\left(1-\left(1+\frac{r}{m}\right)^{-nm}\right)^{-1}$

remaining principal after k payments (of amount C) is

$$\underset{\substack{\uparrow\\ \text{after } k \text{ payments}}}{PV_k} = C\sum_{i=1}^{nm-k}\left(1+\frac{r}{m}\right)^{-i}$$

↗ the value (after k payments) of the remaining $nm-k$ cash-flows (this is what we would reasonably call "remaining value" or "remaining principal")

HW: create amortization schedule

Internal Rate of Return (IRR) / yield:

general cash-flow: given n, C_i, price P, the r that solves

$$PV(r) = \boxed{\sum_{i=1}^{n} \frac{C_i}{(1+r)^i} = P}$$ is called IRR

price of the financial instrument

Sometimes one defines the net-present value $NPV(r) = PV(r) - P$

(then IRR = zero of $NPV(r)$).