Stochastic Methods Lab, Jacobs University Bremen, Fall 2020

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Session 4
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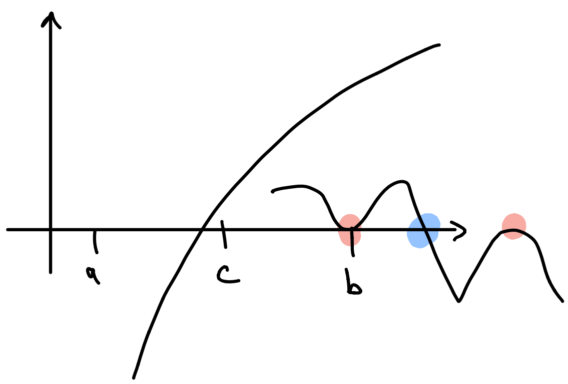
Root Finding Algorithms:

- **Bisection:**



- choose $a<b$ such that $f(a)f(b)<0$

  (if $f(a)f(b)=0$ then we're done: a or b is the root)

- set $c = \frac{a+b}{2}$

  ↳ if $f(c)=0 \Rightarrow$ done

  ↳ if $f(a)f(c)<0 \Rightarrow$ root in $[a,c]$

  ↳ if $f(c)f(b)<0 \Rightarrow$ root in $[c,b]$

- repeat with either $[a,c]$ or $[c,b]$

- Advantages: • robust method, only continuity of f necessary

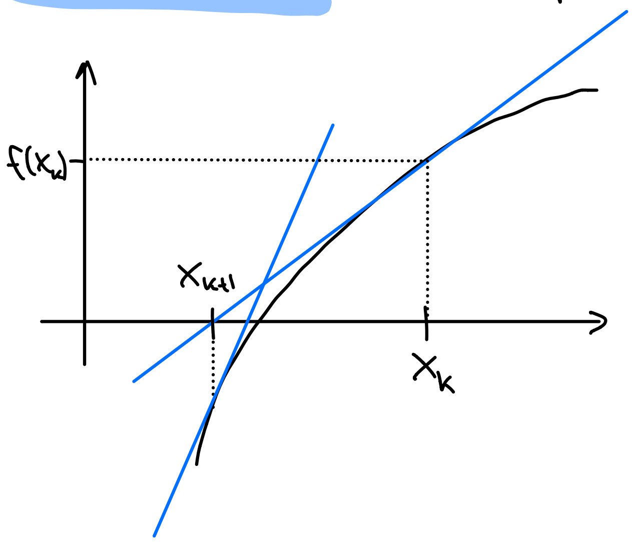
  (except if $f(x) \geq 0 \ \forall x$)

- Disadvantages: • roots where $f(x) \geq 0$ (or ≤ 0) in some region around x cannot be found

  • very slow: error after n+1 steps $\varepsilon_{n+1} = \frac{1}{2}\varepsilon_n$ error after n steps

  $\Rightarrow$ on the r.h.s. we have ε_n^{1} $\Rightarrow$ linear convergence

• $\boxed{\text{Newton's Method}}$ (Newton-Raphson method)



- we start by choosing some x_k

- then $f'(x_k) = \dfrac{f(x_k)}{x_k - x_{k+1}}$

$$\Rightarrow x_k - x_{k+1} = \frac{f(x_k)}{f'(x_k)}$$

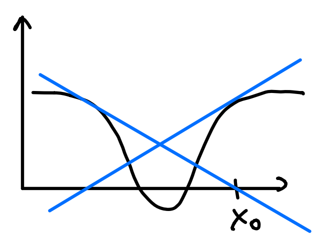
$$\Rightarrow x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

↓

Iteration

- Advantages: fast: quadratic convergence (see below)

- Disadvantages: • need differentiable f
  • need explicit expression for derivative f' (or need extra numerical computation)
  • it might not work!
    ↳ possible problems: - if f'' not continuous then convergence rate will be worse than quadratic
    - $f'(x_k) = 0$ for some k
    - x_0 too far away from the zero
    - cyclic behavior:



What is the rate of convergence?

Use Taylor expansion around x_k:

$$f(z) = f(x_k) + f'(x_k)(z-x_k) + \frac{f''(x_k)}{2}(z-x_k)^2 + \underbrace{O((z-x_k)^3)}_{R \text{ (rest)}}$$

Let z be the root, i.e., $f(z)=0$

$$\Rightarrow \quad 0 = f(x_k) + f'(x_k)(z-x_k) + \frac{f''(x_k)}{2}(z-x_k)^2 + R$$

$$x_k = x_{k+1} + \frac{f(x_k)}{f'(x_k)}$$

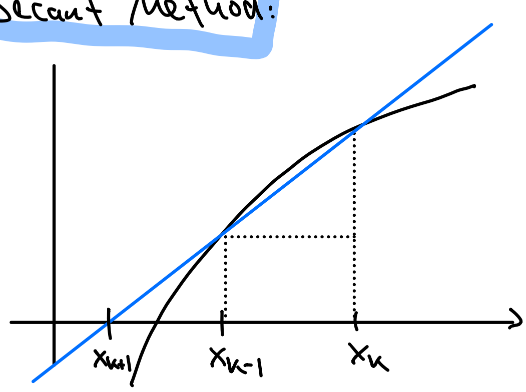
$$\Rightarrow 0 = f(x_k) + f'(x_k)\left(z - x_{k+1} - \frac{f(x_k)}{f'(x_k)}\right) + \frac{f''(x_k)}{2}(z-x_k)^2 + R$$

$$\Rightarrow \quad z - x_{k+1} \approx -\frac{f''(x_k)}{2f'(x_k)}(z-x_k)^2$$

$\Rightarrow$ error after $k+1$ steps $\varepsilon_{k+1} = |z - x_{k+1}| \approx \underbrace{\left|\frac{f''(x_k)}{2f'(x_k)}\right|}_{\text{if this stays indeed bounded}} \varepsilon_k^2$

$\Rightarrow$ quadratic convergence!

- **Secant Method:**



- take secants instead of tangents

intercept thm. (Thales, "Strahlensatz"):

$$\frac{f(x_k)}{x_k - x_{k+1}} = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

$$\Rightarrow x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$ is the iteration

- Advantages: • still relatively fast, rate of convergence is the golden ratio ≈ 1.62
  • derivative not needed

- otherwise very similar to Newton

Pythons built-in fct.: $\boxed{\text{brentq}}$

- combines advantages of several methods
- always converges for continuous functions

$\Rightarrow$ robust and relatively fast