Stochastic Methods Lab, Jacobs University Bremen, Fall 2020
Prof. Dr. Soeren Petrat

Session 6
Sep. 18, 2020

2. Options and Binomial Tree Models

2.1 Option Basics

option: contract (or financial instrument) that depends on the future price of some other underlying asset (most commonly, stocks, which we will focus on)

=> this is called a "derivative" financial instrument

call option: holder can buy underlying asset for price K ("strike price") at time T ("expiration date")

put option: holder can sell underlying asset for price K at time T

+ type/name of stock specify the option contract

Definitions: • price of underlying asset will be called S $(S(t))$

• payoff = value of the option at expiration time T

Ex.: strike price $K = 50\$$

↳ suppose at T, the stock price $S(T) = 60\$$

↳ call option (buy): payoff $= 60\$ - 50\$ = 10\$$ (exercise option)

↳ put option (sell): payoff $= 0\$$ (not exercise option)

Note: the two most common types of option are:

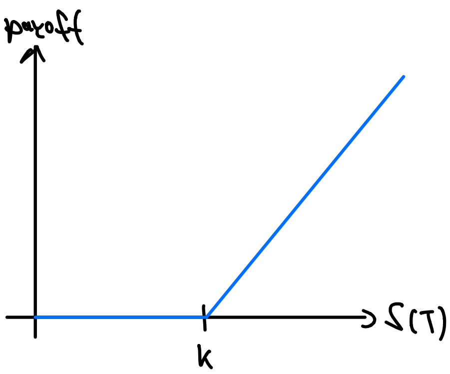
- European options: can be exercised only at expiration date T
- American options: can be exercised at or before expiration T

What are options good for:

- betting / speculation
- can play role of insurance

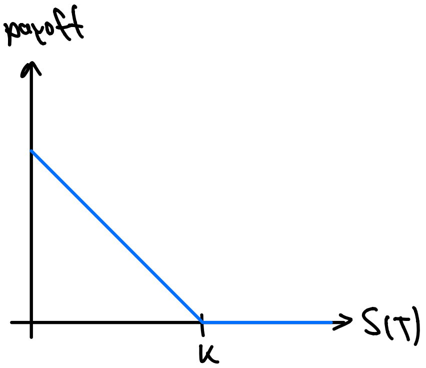
Payoffs:

call



$$\text{payoff } \underbrace{C}_{\text{call payoff}} = \max(0, S(T) - K)$$

put



payoff $\underbrace{P}_{\text{put payoff}} = \max(0, K - S(T))$

$(0 \leq S(T) < \infty)$

note: · buying option: "long position"
· selling option: "short position"

Goal for most of the rest of class:

What is a fair price of an option?

(Is there actually an answer? Stock prices not predictable...)

Assumptions:

- there is a risk-free market, which we take to be a bond market, with risk-free interest rate r, constant in time (e.g., US treasury bonds, or ECB bonds)
- stocks and bonds can be bought and sold unlimitedly and without transaction costs

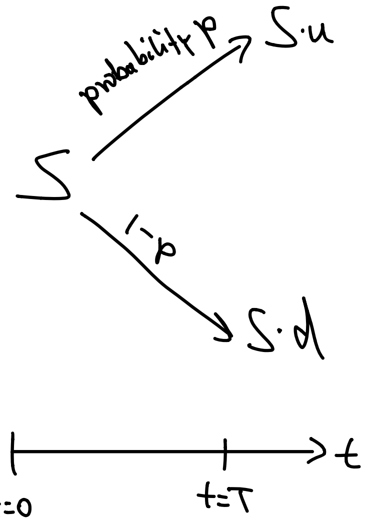
Problem: stock price is uncertain

$\Rightarrow$ $\boxed{\text{we need a probabilistic model for it (for } S(t),\ 0 \le t \le T\text{)}}$

$\boxed{\begin{array}{l}\text{Main idea for "fair pricing":} \\ \quad \text{no opportunity for risk-free profit} \\ \quad\quad = \text{no arbitrage assumption}\end{array}}$

2.2 Binary Model

first, simple model with only 2 possibilities and one time step:



(p, u, d are the parameters of our model)

($d < u$; keep in mind $u > 1$, $d < 1$)

Today, let's look at an example:

without the strategy below

$S = 2500\,\$$, $K = 3000\,\$$, $r = 0$, call

profit for

			seller:	buyer:
$S = 2500\,\$$	$\xrightarrow{p=\frac{1}{2}}$ $4000\,\$$	$\rightarrow$ payoff $= 1000\,\$$	$-500\,\$$	$500\,\$$
	$\xrightarrow{p=\frac{1}{2}}$ $2000\,\$$	$\rightarrow$ payoff $= 0\,\$$	$+500\,\$$	$-500\,\$$

Obvious idea: set price at $C = \frac{1}{2}(1000\,\$) + \frac{1}{2}\,0\,\$ = 500\,\$$

But then one strategy of the seller could be the following:

at $t=0$: sell option (for $500\,\$$), borrow $2000\,\$$ and buy one stock (for $2500\,\$$)

↳ if $S(T) = 4000\,\$$ $\rightarrow$ option will be exercised, seller has to sell stock for $K = 3000\,\$$

$\Rightarrow$ profit: $3000\,\$ - 2000\,\$ = 1000\,\$$

(holder has made $1000\,\$ - 500\,\$ = 500\,\$$)

↳ if $S(T) = 2000\,\$$ $\rightarrow$ option will not be exercised, one could sell stock for $2000\,\$$

$\Rightarrow$ profit: $2000\,\$ - 2000\,\$ = 0\,\$$

(holder has a balance of $-500\,\$$)

$\Rightarrow$ bad option price, since seller can make risk-free profit

$\Rightarrow$ general idea: construct portfolio of stocks and bonds in such a way that the obligation can always be met, and which then mimics the option price,

called "replicating portfolio"