Stochastic Methods Lab, Jacobs University Bremen, Fall 2020

Session 8
Sep. 25, 2020

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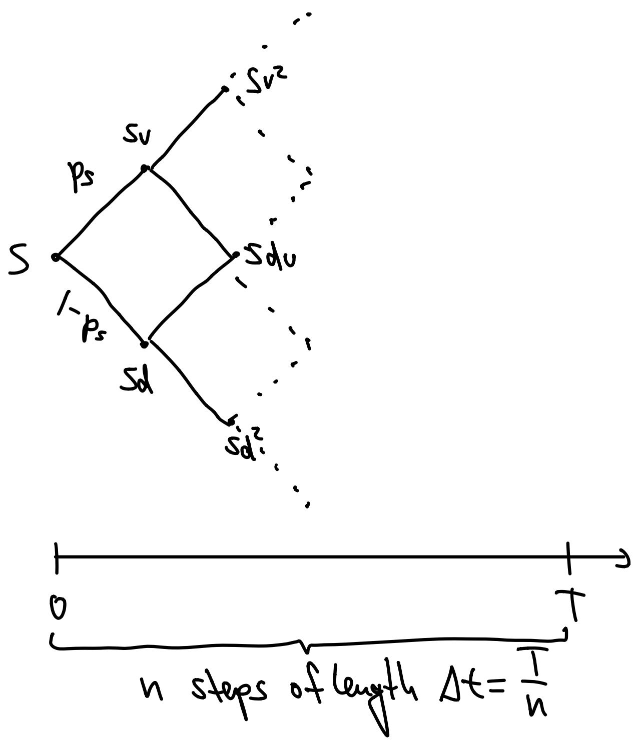
2.3 Binomial Tree Models

repeat binary model with n steps

1. Stock price model:



$\Rightarrow$ stock price after j upward movements after n steps: $S_T(j \text{ up's}) = S u^j d^{n-j}$

probability for j up movements for n steps is $P(j,n) = \underbrace{\binom{n}{j}}\, p_s^j (1-p_s)^{n-j}$

$$= \frac{n!}{(n-j)!\,j!} = \frac{n(n-1)\cdots(n-j+1)}{j!} \quad (\text{"}n \text{ choose } j\text{"})$$

(recall: $(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}$, the binomial theorem)

Is this really a probability?

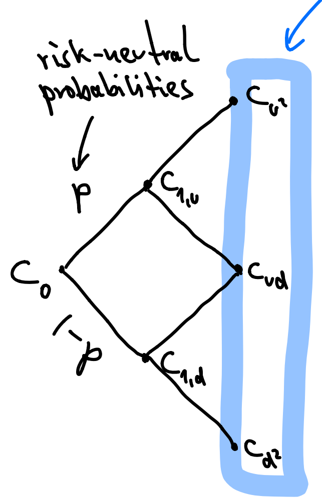
$$\sum_{j=0}^{n} P(j,n) = \sum_{j=0}^{n} \binom{n}{j} p_s^j (1-p_s)^{n-j} = \left(p_s + (1-p_s)\right)^n = 1 \Rightarrow \text{yes (all possibilities add up to 1)}$$

We will come back to reasonable choices of parameters u, d, p_s later.

2. Option Price Model:

here, at expiration, we know the payoff (given the possible $S(T)$)

$n=2$



$$\left.\begin{aligned} C_{u^2} &= \max(0, Su^2 - K) \\ C_{ud} &= \max(0, Sud - K) \\ C_{d^2} &= \max(0, Sd^2 - K) \end{aligned}\right\} \text{for European calls}$$

K = strike price, $p = \dfrac{e^r - d}{u - d}$ ($C_{1,u}, C_{1,d}$ are $\underbrace{\text{"intermediate payoffs"}}_{\text{"option value at step 1"}}$, C_0 = option price)

We know from binary model how to do one step:

$$\Rightarrow C_{1,u} = e^{-r}\left(p\, C_{u^2} + (1-p)\, C_{ud}\right)$$

$$\Rightarrow C_{1,d} = e^{-r}\left(p\, C_{ud} + (1-p)\, C_{d^2}\right)$$

r = interest rate for one step

next step (from 1 to 0):

$$C_0 = e^{-r}\left(p\,C_{1,u} + (1-p)C_{1,d}\right)$$

$$= e^{-2r}\left(p^2 C_{u^2} + p(1-p)C_{ud} + (1-p)p\,C_{ud} + (1-p)^2 C_{d^2}\right)$$

$$= e^{-2r}\left(p^2 C_{u^2} + 2p(1-p)C_{ud} + (1-p)^2 C_{d^2}\right) \rightarrow \text{option price at time/step } 0$$

In this case (European call options without dividend payments) we get the closed-form formula (for n steps):

$$\boxed{C_0 = e^{-nr}\sum_{j=0}^{n}\binom{n}{j}p^j(1-p)^{n-j}\max(0, S u^j d^{n-j} - K)}$$

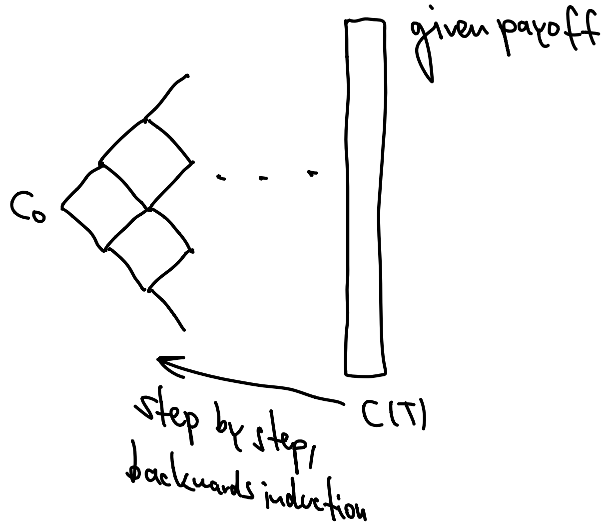
important note: in this notation r is the interest rate for one step; for the period interest rate r_p (annual if T is in years) we have $\boxed{r = r_p\,\Delta t = r_p\frac{T}{n}}$

(so also $p = \dfrac{e^{r_p\frac{T}{n}} - d}{u-d}$)

Note: In the general case and for more complicated models (e.g. puts or dividend payments or discontinuous interest compounding) there might not be closed-form formulas, so it is better to have an algorithm available using "backwards induction" (meaning: start from last column, go to step $n-1$, then $n-2$, ..., until at time 0 you get the result C_0); still based on binomial tree.

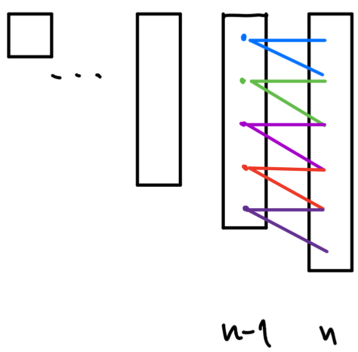
This is a very general and versatile way of option pricing.

For several steps:



Implementation in python:

- use one "for loop" to go through all the steps
- but: computation of vector/array $C(n-1)$ from vector $C(n)$ should be implemented vectorized



recall the notation vector[a:b:increment]

- to store data one could use:
  - one vector (length $n+1$); memory efficient
  - an $(n+1)\times(n+1)$ matrix; if all data is needed, e.g., for visualization

$\rightarrow$ HW 3

2.4 Binomial Tree and Calibration

We want to choose u, d, p_s in such a way that they match a given (observed) expectation value and variance, for large n (many steps).

We had $P(j,n) = \binom{n}{j} p_s^j (1-p_s)^{n-j}$, $S_T(j\ up) = S_0 u^j d^{n-j}$

Now let's put $S_T(j\ up) = S_0 e^{Y_j}$, Y_j = stock's rate of return

$$\Rightarrow Y_j = \ln \frac{S_T(j\ up)}{S_0} = \ln u^j d^{n-j} = \ln \left(\frac{u}{d}\right)^j d^n = j \ln\left(\frac{u}{d}\right) + n \ln d$$

$\ln(ab) = \ln a + \ln b$
$\ln(a^x) = x \ln a$

Next: compute expectation value and variance of Y_j (Y as a fct. of j).

Def.:
- Expectation value of X is $\mathbb{E}(X) = \sum_{j=0}^{n} X_j P(j,n)$
- Variance of X is $\mathrm{Var}(X) = \mathbb{E}(X - \mathbb{E}(X))^2$

A few computational rules:

- $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$, $\mathbb{E}(\lambda X) = \lambda \mathbb{E}(X)$
- $\mathrm{Var}(X) = \mathbb{E}\left(X^2 - 2X\,\mathbb{E}(X) + \mathbb{E}(X)^2\right) = \mathbb{E}(X^2) - 2\,\mathbb{E}(X)\,\mathbb{E}(X) + \mathbb{E}(X)^2$

$$= \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

- $\mathrm{Var}(\lambda x) = \lambda^2 \, \mathrm{Var}(x)$

- $\mathrm{Var}(x+y) = \mathbb{E}((x+y)^2) - \left(\mathbb{E}(x+y)\right)^2$

$$= \mathbb{E}(x^2) + 2\,\mathbb{E}(xy) + \mathbb{E}(y^2) - \left(\mathbb{E}(x)^2 + 2\,\mathbb{E}(x)\mathbb{E}(y) + \mathbb{E}(y)^2 \right)$$

$$= \mathrm{Var}(x) + \mathrm{Var}(y) + 2 \underbrace{\left(\mathbb{E}(xy) - \mathbb{E}(x)\mathbb{E}(y) \right)}$$

$= \mathrm{Cov}(x,y) =$ covariance; does not generally vanish, but is zero for independent x and y

Next: y_j is a linear fct. of j, so we need to compute $\mathbb{E}$ and Var of $x_j = j$ (the identity fct.), i.e., $\underbrace{\mathbb{E}(x) \equiv \mathbb{E}(x_j) \equiv \mathbb{E}(j) \equiv \mathbb{E}(\mathbb{1}) \equiv \mathbb{E}(id)}_{\text{different notations for same object}}$.

$$\mathbb{E}(x) = \sum_{j=0}^{n} j \, P(j,n) = \sum_{j=0}^{n} j \binom{n}{j} p_s^j (1-p_s)^{n-j}$$

$$= (1-p_s)^n \sum_{j=0}^{n} j \binom{n}{j} \left(\frac{p_s}{1-p_s} \right)^j$$

note: $\sum_{j=0}^{n} j \binom{n}{j} z^j$ can be computed by shifting summation indices or via derivatives:

$$z \frac{d}{dz} \underbrace{\sum_{j=0}^{n} \binom{n}{j} z^j}_{=(1+z)^n} = z \sum_{j=0}^{n} j \binom{n}{j} z^{j-1} = \sum_{j=0}^{n} j \binom{n}{j} z^j$$

$$\Longrightarrow \sum_{j=0}^{n} j \binom{n}{j} z^j = z \frac{d}{dz} (1+z)^n = z n (1+z)^{n-1}$$

$$\Rightarrow \mathbb{E}(x) = (1-p_s)^n \left(\frac{p_s}{1-p_s}\right) n \underbrace{\left(1+\frac{p_s}{1-p_s}\right)^{n-1}}_{\left(\frac{1}{1-p_s}\right)^{n-1}} \qquad \left(z=\frac{p_s}{1-p_s}\right)$$

$$= \left(\frac{p_s}{1-p_s}\right) n \, (1-p_s)$$

$$= n p_s$$

by similar computation: $\mathbb{E}(x^2) = n p_s \left[(n-1) p_s + 1 \right]$

$$\Rightarrow \mathrm{Var}(x) = \mathbb{E}(x^2) - \mathbb{E}(x)^2 = n p_s - n p_s^2 = n p_s (1-p_s)$$

Now we can compute $\mathbb{E}$ and Var of $Y_j = j \ln\left(\frac{v}{d}\right) + n \ln d$

We find:

- $\mathbb{E}(Y) = \mathbb{E}\left(j \ln\left(\frac{v}{d}\right) + n \ln d\right) = \ln\frac{v}{d}\, \mathbb{E}(j) + n \ln d = \left(\ln\frac{v}{d}\right) n p_s + n \ln d$

- $\mathrm{Var}(Y) = \mathrm{Var}\left(j \ln\frac{v}{d} + n \ln d\right) \underset{\substack{\uparrow \\ \mathrm{Cov}(\mathrm{const}, x) = 0 \\ \mathrm{Var}(\mathrm{const}) = 0}}{=} \left(\ln\frac{v}{d}\right)^2 \mathrm{Var}(j) = \left(\ln\frac{v}{d}\right)^2 n p_s (1-p_s)$

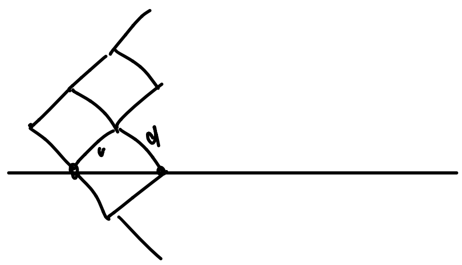
Next, we want to match $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$ to given values:

$$\mathbb{E}(Y_j) \xrightarrow{n\to\infty} \mu T \quad , \quad \mathrm{Var}(Y_j) \xrightarrow{n\to\infty} \sigma^2 T$$

μ is called mean-value

σ is called volatility

there is one more sensible condition: $ud = 1$



same value for stock price if $ud = 1$ on all horizontal lines

Then
- $\mathbb{E}(Y) = \left(\ln \frac{u}{d}\right) n p_s + n \ln d = 2(\ln u) n p_s - n \ln u = n(\ln u)(2p_s - 1)$
- $\mathrm{Var}(Y) = \left(\ln \frac{u}{d}\right)^2 n p_s (1-p_s) = 4 (\ln u)^2 n p_s (1-p_s)$

Still there are several possible choices for u and p_s, a common one is

$$\boxed{p_s = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{T}{n}}, \quad u = e^{\sigma\sqrt{\frac{T}{n}}}} \qquad \left(d = e^{-\sigma\sqrt{\frac{T}{n}}}\right)$$

Check that they indeed give the right $\mathbb{E}$ and Var for large n:

- $\mathbb{E}(Y_j) = n\sigma\sqrt{\frac{T}{n}}\,\frac{\mu}{\sigma}\sqrt{\frac{T}{n}} = \mu T$
- $\mathrm{Var}(Y_j) = 4\left(\sigma\sqrt{\frac{T}{n}}\right)^2 n \left(\frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{T}{n}}\right)\left(\frac{1}{2} - \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{T}{n}}\right)$

$$= \sigma^2 T \left(1 + \frac{\mu}{\sigma}\sqrt{\frac{T}{n}}\right)\left(1 - \frac{\mu}{\sigma}\sqrt{\frac{T}{n}}\right)$$

$$= \sigma^2 T - \underbrace{\sigma^2 T\left(\frac{\mu}{\sigma}\sqrt{\frac{T}{n}}\right)^2}_{\to 0 \text{ as } n\to\infty} \xrightarrow{n\to\infty} \sigma^2 T$$

Note again: μ and σ can be read off from real data (we will talk about this later),

therefore we want to choose $u, d, \underbrace{p_S}_{\text{not needed for option pricing}}$ depending on μ, σ.

Note: For the choice above u does not depend on μ, only on σ

$\Rightarrow$ our option price is independent of μ!