2.5 Central Limit Theorem

Recall that we encountered the binomial distribution:

- "up" with probability p
- "down" with probability $1-p$

total number of steps; probability for "up"

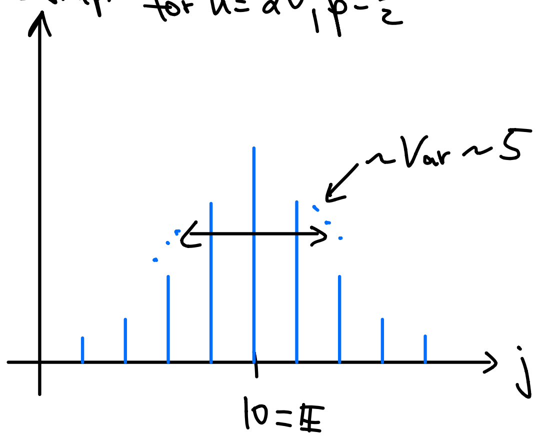
probability for j "up"s is $b(j;n,p) = \binom{n}{j} p^j (1-p)^{n-j}$ $\quad \left(\binom{n}{j} = \frac{n!}{(n-j)!\,j!} \right)$

number of "up"s

recall: • $\mathbb{E}(j) = np$

• $\mathrm{Var}(j) = np(1-p)$

$b(j;n,p)$ for $n=20$, $p=\frac{1}{2}$



Note: in order to compare distributions (here, pictures for different n), we need to center, and normalize the variance

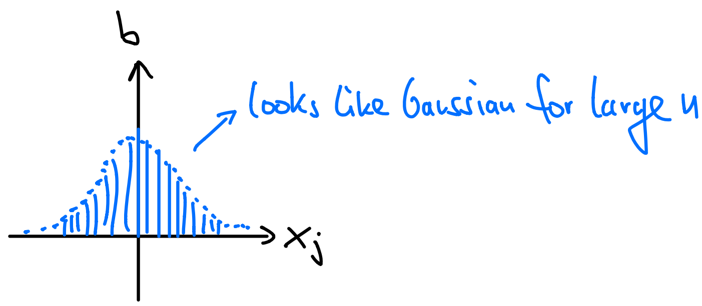
- centering: introduce $y_j = j - \mathbb{E}(j) = j - np$, such that

$$\mathbb{E}(y_j) = \mathbb{E}(j - np) = \mathbb{E}(j) - np = 0$$

- normalize variance (plus centering): $x_j = \dfrac{j - np}{\sqrt{np(1-p)}}$

$$\Rightarrow \underset{\uparrow\, \mathrm{Var}(\lambda X) = \lambda^2 \mathrm{Var}(X)}{\mathrm{Var}(x_j)} = \frac{1}{np(1-p)} \mathrm{Var}(j - np) = 1$$

$$\Rightarrow \quad j = \sqrt{np(1-p)}\, x_j + np$$



next: look at cumulative distribution, meaning the probability for A or fewer "up"s.

It is given by $\displaystyle\sum_{j=0}^{A} b(j;n,p)\, \underbrace{\Delta j}_{=1 = \text{distance between } j\text{'s}}$

With the change of variables above, $j = \sqrt{np(1-p)}\, x_j + np$ and so $\Delta j = \sqrt{np(1-p)}\, \Delta x_j$,

so we should get (let A also depend on n)

$$\sum_{j=0}^{A_n} b(j;n,p)\, \Delta j = \sum_{x = \frac{-np}{\sqrt{np(1-p)}}}^{\frac{A_n - np}{\sqrt{np(1-p)}}} b\left(\sqrt{np(1-p)}\, x + np, n, p\right) \sqrt{np(1-p)}\, \Delta x$$

$\tilde{A} \leftarrow$ if A_n is chosen nicely (e.g., $A_n = np + \tilde{A}\sqrt{np(1-p)}$)

$$\xrightarrow[\text{should}]{n \to \infty} \int_{-\infty}^{\tilde{A}} \varphi(x)\, dx$$

$\varphi \leftarrow$ some limiting fct.

Such a convergence result is called **Central Limit Theorem (CLT)**

So for the binomial distribution, we get:

$$\boxed{\begin{array}{l} \sqrt{np(1-p)}\; b\left(\sqrt{np(1-p)}\,x + np, n, p\right) \xrightarrow{n\to\infty} \varphi(x) \quad \text{pointwise} \\[2ex] \text{with} \quad \varphi(x) = \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}_{\text{normalized Gaussian}} = \underbrace{\mathcal{N}}_{\text{"normal distribution"}}(\overset{\text{mean}}{0}, \underset{\text{variance}}{1}) \end{array}}$$

Remarks:

- here we get pointwise convergence, but generally the CLT gives us convergence in the sense of cumulative distribution functions

- Let's check normalization:

$$\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \right)^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; e^{-\left(\frac{x^2+y^2}{2}\right)}$$

polar coordinates
$x^2+y^2=r^2$
$dx\,dy = r\,dr\,d\varphi$

$$= \frac{1}{2\pi} \int_0^{\infty} dr \int_0^{2\pi} d\varphi \; r e^{-\frac{r^2}{2}}$$

$$= \int_0^{\infty} dr \; r e^{-\frac{r^2}{2}}$$

$$= -e^{-\frac{r^2}{2}} \Big|_0^{\infty}$$

$$= 1$$

- one can also check that indeed $\mathbb{E}(x) = 0$ and $\mathrm{Var}(x) = 1$