2.6 Black-Scholes Formula

recall the exact formula for the price of European call options

$$C = e^{-rT} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \max(0, S u^j d^{n-j} - K) \quad (K \text{ strike price}, r \text{ period interest rate})$$

$$= e^{-rT} \mathbb{E}(\text{payoff})$$

↳ under binomial distribution

Many terms in the sum are 0; payoff $\neq 0$ if $S u^j d^{n-j} - K > 0$

$$\Rightarrow \text{need } u^j d^{n-j} > \frac{K}{S} \Rightarrow \left(\frac{u}{d}\right)^j > \frac{K}{S d^n} \Rightarrow j > \frac{\ln\left(\frac{K}{S d^n}\right)}{\ln\left(\frac{u}{d}\right)} = \frac{\ln\left(\frac{K}{S}\right) - n \ln d}{\ln\left(\frac{u}{d}\right)} =: A_n$$

$$\Rightarrow C = e^{-rT} \sum_{j=A_n}^{n} \binom{n}{j} p^j (1-p)^{n-j} (S u^j d^{n-j} - K)$$

$$= S \sum_{j=A_n}^{n} \binom{n}{j} \left(p u e^{-r\frac{T}{n}}\right)^j \left((1-p) d e^{-r\frac{T}{n}}\right)^{n-j} - K e^{-rT} \sum_{j=A_n}^{n} \binom{n}{j} p^j (1-p)^{n-j}$$

recall $p = \frac{e^{r\frac{T}{n}} - d}{u-d} \Rightarrow (1-p) = \frac{u - d - e^{r\frac{T}{n}} + d}{u-d} = \frac{u - e^{r\frac{T}{n}}}{u-d}$

$$\Rightarrow (1-p) d e^{-r\frac{T}{n}} = \frac{u - e^{r\frac{T}{n}}}{u-d} d e^{-r\frac{T}{n}} = \frac{u d e^{-r\frac{T}{n}} - d + u - u}{u-d} = 1 - p u e^{-r\frac{T}{n}}$$

$$\Rightarrow C = S \sum_{j=A_n}^{n} b\left(j, n, p u e^{-r\frac{T}{n}}\right) - K e^{-r\frac{T}{n}} \sum_{j=A_n}^{n} b(j, n, p)$$

Next: use our calibration $u = e^{\sigma\sqrt{\frac{T}{n}}}$, $d = \frac{1}{u} = e^{-\sigma\sqrt{\frac{T}{n}}}$

$$\Rightarrow A_n = \frac{\ln\left(\frac{K}{S}\right) - n\ln d}{\ln\left(\frac{u}{d}\right)} = \frac{\ln\left(\frac{K}{S}\right) - n\ln e^{-\sigma\sqrt{\frac{T}{n}}}}{\ln e^{2\sigma\sqrt{\frac{T}{n}}}} = \frac{\ln\left(\frac{K}{S}\right) + \sigma\sqrt{T}\sqrt{n}}{2\sigma\sqrt{\frac{T}{n}}} = \frac{\ln\left(\frac{K}{S}\right)}{2\sigma\sqrt{T}}\sqrt{n} + \frac{1}{2}n$$

$$\Rightarrow p = \frac{e^{r\frac{T}{n}} - d}{u-d} = \frac{e^{r\frac{T}{n}} - e^{-\sigma\sqrt{\frac{T}{n}}}}{e^{\sigma\sqrt{\frac{T}{n}}} - e^{-\sigma\sqrt{\frac{T}{n}}}} \overset{\text{Taylor}}{=} \frac{1 + r\frac{T}{n} + O(n^{-2}) - \left[1 - \sigma\sqrt{\frac{T}{n}} + \frac{1}{2}\sigma^2\frac{T}{n} + O(n^{-\frac{3}{2}})\right]}{1 + \sigma\sqrt{\frac{T}{n}} + \frac{1}{2}\sigma^2\frac{T}{n} + O(n^{-\frac{3}{2}}) - \left[1 - \sigma\sqrt{\frac{T}{n}} + \frac{1}{2}\sigma^2\frac{T}{n} + O(n^{-\frac{3}{2}})\right]}$$

$$= \frac{\sigma\sqrt{\frac{T}{n}} + \left(r - \frac{\sigma^2}{2}\right)\frac{T}{n} + O(n^{-\frac{3}{2}})}{2\sigma\sqrt{\frac{T}{n}} + O(n^{-\frac{3}{2}})}$$

$$= \frac{1}{2}\left(1 + \frac{\left(r - \frac{\sigma^2}{2}\right)}{\sigma}\sqrt{\frac{T}{n}} + \underbrace{O(n^{-1})}_{= \frac{O(n^{-\frac{3}{2}})}{2\sigma\sqrt{T}n^{-\frac{1}{2}}}}\right)$$

Then the integral boundary from the CLT is

$$\hat{A} = \lim_{n\to\infty} \frac{A_n - np}{\sqrt{np(1-p)}} = \lim_{n\to\infty} \frac{\frac{\ln\left(\frac{K}{S}\right)}{2\sigma\sqrt{T}}\sqrt{n} + \frac{1}{2}n - \left[\frac{n}{2} + \frac{n}{2}\frac{\left(r - \frac{\sigma^2}{2}\right)}{\sigma}\sqrt{\frac{T}{n}} + O(n^{-1})\right]}{\underbrace{\sqrt{n}\left(\frac{1}{2} + \frac{\text{const}}{\sqrt{n}} + O(n^{-1})\right)^{\frac{1}{2}}\left(\frac{1}{2} - \frac{\text{const}}{\sqrt{n}} + O(n^{-1})\right)^{\frac{1}{2}}}_{= \sqrt{n}\sqrt{\frac{1}{4} + O(n^{-1})} = \sqrt{n}\,\frac{1}{2} + O(n^{-\frac{1}{2}})}}$$

$$= \frac{\ln\left(\frac{K}{S}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

Now applying the CLT gives us: (after some more computation)

$$\boxed{C = S\,\Phi(x) - Ke^{-rT}\,\Phi(x - \sigma\sqrt{T}), \text{ the Black-Scholes formula,}}$$

with cumulative normal distribution $\Phi(x) = \int_{-\infty}^{x} \underbrace{\varphi(y)}_{\text{normal Gaussian}}\,dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy$

and $x = \dfrac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$

Let's check: $\hat{A} = \dfrac{\ln\left(\frac{K}{S}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = -\left(\dfrac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$

$$= -\left(x - \sigma\sqrt{T}\right)$$

Then: $Ke^{-rT} \sum_{j=A_n}^{n} b(j;n,p) \xrightarrow[\text{CLT}]{n\to\infty} Ke^{-rT} \int_{\hat{A}}^{\infty} \varphi(y)\,dy = Ke^{-rT} \int_{-(x-\sigma\sqrt{T})}^{\infty} \varphi(y)\,dy$

$$= Ke^{-rT} \int_{-\infty}^{x-\sigma\sqrt{T}} \varphi(y)\,dy$$

$$= Ke^{-rT}\,\Phi(x - \sigma\sqrt{T})$$

Similar for the first summand, but we need to use approximations for $pue^{-r\frac{T}{n}}$.

2.7 Convergence Rates

Consider some sequence $C_n \xrightarrow{n\to\infty} C$, e.g., C_n = option price for n step binomial tree, and

C = price from Black-Scholes

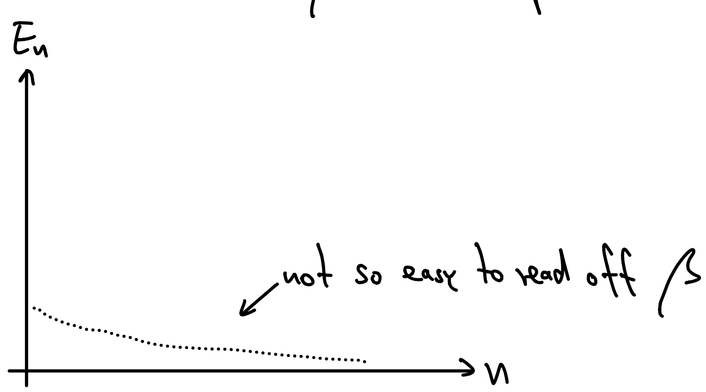
It is very important to know how fast the convergence is.

Often, the convergence goes like a power law: $E_n = |C_n - C| \approx A n^{-\beta}$ for large n

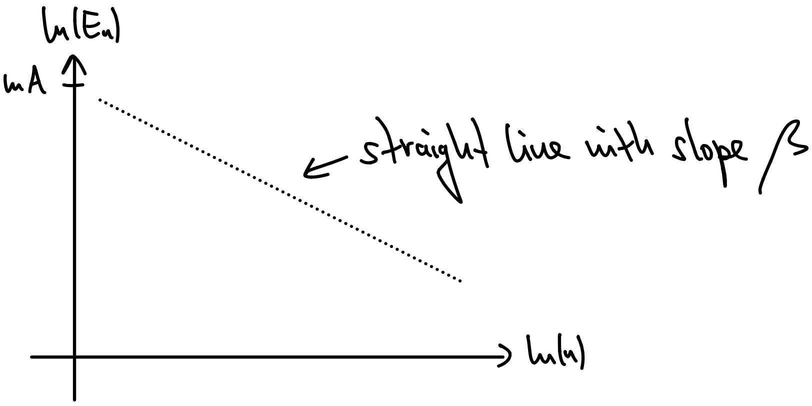
β is called rate of convergence

Remarks:
- different β can make a huge difference (e.g. linear vs. quadratic)
- if limit C is unknown one could, e.g., look at $N >> n$ and then $|C_n - C_N|$, or $\varepsilon_n = |C_{n+1} - C_n|$

How can we read off β from a plot?



better: $\ln E_n = \ln A n^{-\beta} = \ln A - \beta \ln n$



python: loglog(n, E_n) produces such plots