Stochastic Methods Lab, Jacobs University Bremen, Fall 2020
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Session 11
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2.8 Monte-Carlo Method

idea: use random samplings to approximate expectation values

Ex.: binomial tree model for option price of European calls:

$$C = \sum_{j=0}^{n} b(j,n,p) \underbrace{e^{-rT} \max(S u^j d^{n-j} - K, 0)}_{= f(j,n)} = \mathbb{E}_{\text{bin.}}(f)$$

Monte-Carlo: take m samples $j_1, \dots, j_m$ from bin. distribution ($m \ll n$) and compute

$\frac{1}{m} \sum_{k=1}^{m} f(j_k, n)$, the empirical mean, or sample average.

The law of large numbers says that $\boxed{\frac{1}{m} \sum_{k=1}^{m} f(j_k, n) \xrightarrow{m \to \infty} \underbrace{\mathbb{E}(f)}_{\text{theoretical expectation}}}$

Idea/hope of Monte-Carlo method:

- time efficient / fast, since only $m \ll n$ steps are necessary to compute a good approximation
- interesting idea: use randomness to approximate a deterministic quantity

Summary:

- skills:
  - git
  - python / scipy: basics, vector-based coding, timing, plotting, csv files

- finance:
  - cash flows, interest compounding
  - bonds
  - option (European and American calls and put)
  - option pricing with binomial trees (important concepts: $\underbrace{\text{no arbitrage}}_{\text{no risk-free profit}}$, replicating portfolio)
    - ↳ binomial tree implemented in python
    - ↳ explicit formula for European calls
    - ↳ Black-Scholes formula for European calls
    - ↳ put-call parity (to compute put price, given call price): $\underset{\text{call prize}}{C} - \underset{\text{put prize}}{P} = S - Ke^{-rT}$

- numerical methods:
  - root finding
  - how to find convergence rates
  - QQ plots
  - Monte-Carlo

- math:
  - Taylor expansion
  - binomial and normal distribution
  - CLT

Focus of second half of class: - numerical methods
- option pricing with continuous stochastic processes

3. Continuous Time Models

3.1 Brownian Motion

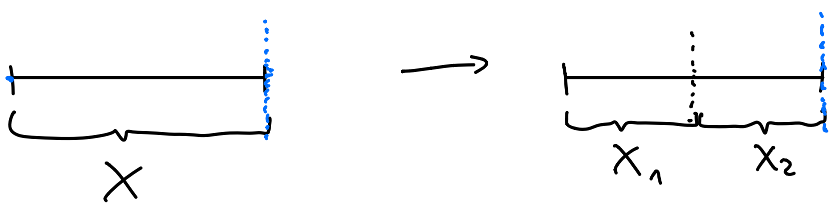
Motivation: Let us consider the normal distribution with mean 0 and variance 1:

$$\mathcal{N}(\underset{\text{mean}}{0}, \underset{\text{standard deviation}}{1}) \qquad (\mathcal{N}(\mu, \sigma) \text{ for mean } \mu, \text{ std. deviation } \sigma)$$

Importance of the normal distribution comes from the Central Limit Theorem (CLT)

Now consider a random variable X that is normally distributed: $X \underset{\text{is distributed according to}}{\sim} \mathcal{N}(0,1)$

Now consider splitting it into two: $X = X_1 + X_2$, s.t. X_1 and X_2 independent and same distribution



Note: $0 = \mathbb{E}(X) = \mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) \overset{X_1 \text{ and } X_2 \text{ same distribution}}{=} 2\,\mathbb{E}(X_1) \Rightarrow X_1$ has expectation 0

How does the variance behave?

$$1 = \mathrm{Var}(X) = \mathrm{Var}(X_1 + X_2) \overset{X_1 \text{ and } X_2 \text{ independent}}{=} \mathrm{Var}(X_1) + \mathrm{Var}(X_2) \overset{\text{same dist.}}{=} 2\,\mathrm{Var}(X_1) \Rightarrow \mathrm{Var}(X_1) = \tfrac{1}{2}$$

$\Rightarrow X_1$ is distributed according to $\underbrace{\mathcal{N}(0,\tfrac{1}{\sqrt{2}}) = \tfrac{1}{\sqrt{2}}\,\mathcal{N}(0,1)}$ $\leftarrow$ $Var(\frac{1}{\sqrt{2}}X) = \frac{1}{2}Var(X)$

both are normal distributions with mean 0 and variance $\frac{1}{2}$,
so they are the same

for n steps: $\quad x_1\ x_2\ x_3 \dots$

$$1 = Var(X) = Var\Big(\sum_{i=1}^{n} X_i\Big) = \sum_{i=1}^{n} Var(X_i) = n\,Var(X_1)$$

$$\Rightarrow Var(X_1) = \frac{1}{n}$$

$$\Rightarrow \text{each } X_i \sim \mathcal{N}\Big(0,\frac{1}{\sqrt{n}}\Big) = \frac{1}{\sqrt{n}}\,\mathcal{N}(0,1)$$

or, calling $\frac{1}{n} = \Delta t$: $X_i \sim \sqrt{\Delta t}\,\mathcal{N}(0,1)$

This motivates the following definition:

<u>Def.</u>: A stochastic process $\overset{\in \mathbb{R}}{t} \longmapsto W(t)$ for $t \geq 0$ is called $\boxed{\text{Brownian Motion (BM)}}$ ← for fixed t, $W(t)$ is a random variable

or $\boxed{\text{Wiener process}}$ if:

a) $W(0) = 0$ (convention)

b) each realization is continuous,

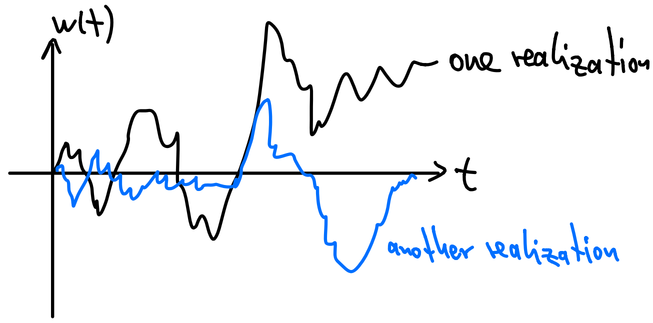
c) for any $0 \leq s_1 < s_2 < t_1 < t_2$ the increments

$W(t_2) - W(t_1)$ and $W(s_2) - W(s_1)$ are independent,

d) $W(t_2) - W(t_1)$ is distributed according to $\sqrt{t_2 - t_1}\,\mathcal{N}(0,1)$ for all $0 \leq t_1 < t_2$.

Note: one can indeed show that such a process exists and is unique

More pictures in python next time:



Is that a good model for stock prices?

No: • BM can become negative

• parameters similar to μ and σ in calibrated binomial tree are missing

Solution (better stock price model):

We use Geometric Brownian Motion (GBM): $\boxed{S(t) = S(0)\, e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}.}$

It turns out (see next HW sheet) that the calibrated binomial tree model converges for $n \to \infty$ indeed to GBM!