Stochastic Methods Lab, Jacobs University Bremen, Fall 2020

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3.2 Stochastic Integrals

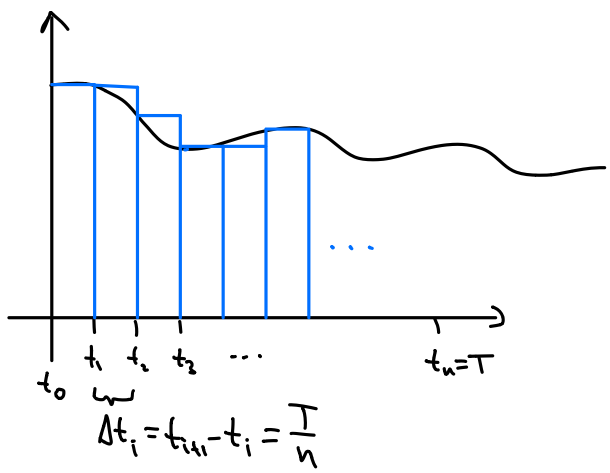
Later: want PDE with randomness = stochastic partial differential equation = SPDE

to model stock market: $dX = f\,dt + g\,dW$, W Brownian motion

In order to make sense of such an equation, we need to know how to define $\int g\,dW$ (bc. $\frac{dW}{dt}$ does not make sense: BM is not differentiable)

Recall Riemann sum for Riemann integral:

$$\int_0^T f(t)\,dt = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\,\Delta t_i$$



There are two kinds of stochastic integrals:

$\boxed{\text{Itô-integral:}}$

defined analogously to Riemann sum:

$$\int_0^T f(t)\,dW(t) := \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\,\Delta W_i \quad \text{with} \quad \overbrace{\Delta W_i}^{\text{random variable}} = W(t_{i+1}) - W(t_i) \overset{\substack{\text{distributed like}\\ \downarrow}}{\sim} \sqrt{t_{i+1}-t_i}\,\mathcal{N}(0,1)$$

$$= \sqrt{\Delta t}\,\mathcal{N}(0,1)$$

Ex.: integrate Brownian motion against itself: $\int_0^T W(t)\,dW(t) = \int_0^T W\,dW$

Note: If $W(t)$ were differentiable, we could use the chain rule $dW = \frac{dW}{dt}dt$

$$\Rightarrow \int_0^T W(t)\,dW(t) = \int_0^T W(t)\,\frac{dW(t)}{dt}\,dt = \frac{1}{2}\int_0^T \frac{d}{dt}\left(W(t)^2\right)dt = \frac{1}{2}W(T)^2 - \frac{1}{2}\underbrace{W(0)^2}_{=0}$$

But $W(t)$ is not differentiable!

Let us compute "by hand":

$$\int_0^T W(t)\,dW(t) := \lim_{n\to\infty}\sum_{i=0}^{n-1} W(t_i)\,\Delta W_i = \lim_{n\to\infty}\sum_{i=0}^{n-1} \underbrace{W(t_i)\left(W(t_{i+1}) - W(t_i)\right)}$$

$$= W(t_i)\,W(t_{i+1}) - W(t_i)^2$$

$$= \frac{1}{2}\left[W(t_{i+1})^2 - W(t_i)^2 - \underbrace{\left(W(t_{i+1}) - W(t_i)\right)^2}_{=W(t_{i+1})^2 - 2W(t_{i+1})W(t_i) + W(t_i)^2}\right]$$

$$\Rightarrow \int_0^T W(t)\,dW(t) = \frac{1}{2}\lim_{n\to\infty}\underbrace{\sum_{i=0}^{n-1}\left(W(t_{i+1})^2 - W(t_i)^2\right)}_{} - \frac{1}{2}\lim_{n\to\infty}\sum_{i=0}^{n-1}\Delta W_i^2$$

$$= W(T)^2 - W(0)^2 = W(T)^2 \quad \text{(telescope sum)}$$

$$(W_1^2 - W_0^2) + (W_2^2 - W_1^2) + (W_3^2 - W_2^2) + \dots (W_n^2 - W_{n-1}^2) = W_n^2 - W_0^2$$

Question: How is $\sum_{i=0}^{n-1}\Delta W_i^2$ distributed?

We know that $\mathbb{E}(\Delta W_i^2) = \Delta t = \frac{T}{n}$ (recall that $\Delta W \sim \sqrt{\Delta t}\,\mathcal{N}(0,1)$)

$$\mathbb{E}(\Delta W_i^2) = \mathrm{Var}(\Delta W_i) = \mathbb{E}(\Delta W_i^2) - \underbrace{\mathbb{E}(\Delta W_i)^2}_{=0}$$

$$\Rightarrow \mathbb{E}\left(\sum_{i=0}^{n-1}\Delta W_i^2\right) = \sum_{i=0}^{n-1}\frac{T}{n} = T$$

What about variance?

$$\mathrm{Var}\left(\sum_{i=0}^{n-1}\Delta W_i^2\right) = \mathbb{E}\left(\sum_{i,j=0}^{n-1}\Delta W_i^2\Delta W_j^2\right) - \underbrace{\mathbb{E}\left(\sum_{i=0}^{n-1}\Delta W_i^2\right)^2}_{T^2}$$

$$= \underbrace{\mathbb{E}\left(\sum_{i=0}^{n-1}\Delta W_i^4\right)}_{} + \underbrace{\mathbb{E}\left(\sum_{i\neq j}\Delta W_i^2\Delta W_j^2\right)}_{} - T^2$$

One can compute: $\approx \sum_{i=0}^{n-1}\frac{T^2}{n^2} = \mathcal{O}\left(\frac{1}{n}\right)$

ΔW_i and ΔW_j independent!

$$= \sum_{i\neq j}\mathbb{E}(\Delta W_i^2\Delta W_j^2) = n(n-1)\frac{T^2}{n^2} = T^2 + \mathcal{O}\left(\frac{1}{n}\right)$$

$$= \mathcal{O}\left(\frac{1}{n}\right)$$

$\Rightarrow$ Variance vanishes in the limit $n\to\infty$ $\Rightarrow \sum_{i=0}^{n-1}\Delta W_i^2 = T$, a constant

$\Rightarrow$ a deterministic process

$\boxed{\text{Conclusion: } \int_0^T W(t)\,dW(t) = \frac{1}{2}W(T)^2 - \frac{1}{2}T}$ (different from usual integral!)

Another possible integral is the

$\boxed{\text{Stratonovich Integral:}}$

$$\int_0^T f(t) \circ dW(t) = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i^*)\,\Delta W_i \qquad \text{with } t_i^* = \frac{t_{i+1}+t_i}{2}$$

notation to differentiate it from Itô integral

Let's consider same example as before:

$$\int_0^T W(t) \circ dW(t) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \underbrace{W(t_i^*)\left(W(t_{i+1}) - W(t_i)\right)}$$

$$\frac{1}{2}\left[W(t_{i+1})^2 - W(t_i)^2 + \left(W(t_i^*) - W(t_i)\right)^2 - \left(W(t_{i+1}) - W(t_i^*)\right)^2 \right]$$

Similar to before: $\cdot\ \mathbb{E}\left(\left(W(t_i^*) - W(t_i)\right)^2\right) = t_i^* - t_i = \frac{t_{i+1}+t_i}{2} - t_i = \frac{t_{i+1}-t_i}{2} = \frac{\Delta t}{2}$

$\cdot\ \mathbb{E}\left(\left(W(t_{i+1}) - W(t_i^*)\right)^2\right) = t_{i+1} - \left(\frac{t_{i+1}+t_i}{2}\right) = \frac{\Delta t}{2}$

and variances vanishes

$$\Rightarrow \int_0^T W(t) \circ dW(t) = \frac{1}{2}W(T)^2 + \frac{T}{2} - \frac{T}{2} = \frac{1}{2}W(T)^2$$

(as we would expect from regular calculus)

In comparison:

- Stratonovich: • much "nicer", properties more similar to usual integration
  - but in each step function is evaluated in between t_{i+1} and t_i

    ↳ undesirable for some SPDEs

- Itô: • technically a bit "harder" to handle, results different from regular calculus
  - but, at each t_i, f is evaluated and an increment is added

    ↳ this is what we want for stock market SPDE later

A few notes on python implementation of BM:

- BM: $W_0 = 0$

  $W_1 = \sqrt{\Delta t} \cdot$ sample from $\mathcal{N}(0,1)$

  $W_2 = W_1 + \sqrt{\Delta t} \cdot$ sample from $\mathcal{N}(0,1)$

  ...

in python: $dW = \text{normal}(0, 1, \text{size}=n) \cdot \sqrt{\Delta t} \longrightarrow$ vector

$W = \text{cumsum}(dW)$ (= vector containing entries of cumulative sum)

$W = \underbrace{\text{r\_}[0, W]}$ (add 0 at time 0)

r_[a,b] appends row vector b to vector a

$(a = (a_0, a_1, \dots, a_k), b = (b_0, b_1, \dots, b_\ell) \Rightarrow \text{r\_}[a,b] = (a_0, a_1, \dots, a_k, b_0, b_1, \dots, b_\ell))$

- ensemble of BMs:

# of time steps

$dW = \text{normal}(0, 1, \text{size} = (M, N)) \cdot \sqrt{\Delta t}$

# of samples

$W = \text{cumsum}(dW, \text{axis}=1)$

cumulative sum over row entries (axis=0 would be columns)

add zero vector

Note: • similarly one can use, e.g., mean(W, axis=0)

mean over samples

transpose to plot rows

• W[:10:,:] selects 10 sample paths (plot(t, W[:10:,:].T))

- seed(k) (k some number) gives you the same samples (same random numbers)