Stochastic Methods Lab, Jacobs University Bremen, Fall 2020

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Session 15
Oct. 10, 2020

4. Black-Scholes Equation and Finite Difference Schemes

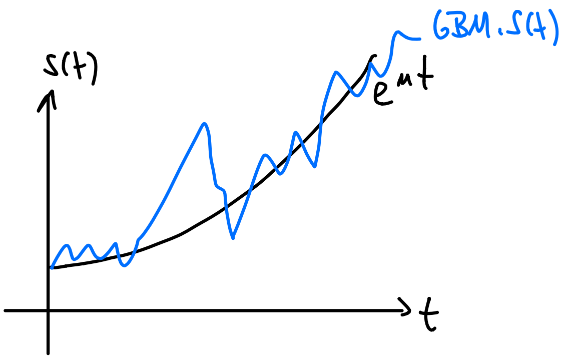
4.1 Derivation of the Black-Scholes Equation

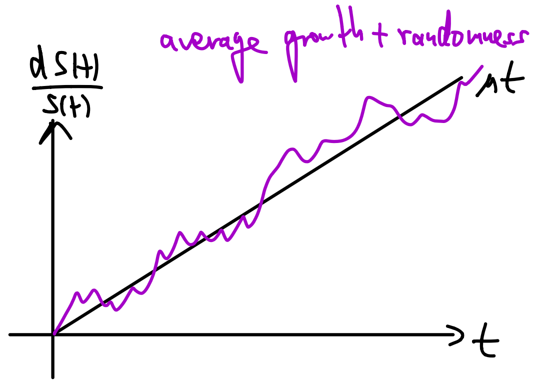
We assume that the stochastic process for stock prize development is geometric Brownian motion (GBM):

$$dS = \mu S\, dt + \sigma S\, dW$$

This means: Stock's rate of return $\frac{dS}{S} = \mu dt + \sigma dW$ has expectation μdt and variance $\sigma^2 dt$.

To visualize:





"Stocks behave like regular cash-flows/bonds $\left(\frac{dX}{X} = r dt\right)$, but with risk ($\sigma dW$ term)."

Recall: From Itô's lemma we found that $S(t) = S_0\, e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$.

Now: Option price C is a function of $S(t)$ and t, so $C = C(S(t), t) = C(x,t)\big|_{x=S(t)}$

Recall Itô's lemma: If $X(t)$ is sol. to $dX = f\,dt + g\,dW$, and $F(X(t),t)$, then

$$dF = \left[\frac{\partial F}{\partial t} + f \frac{\partial F}{\partial x} + \frac{1}{2} g^2 \frac{\partial^2 F}{\partial x^2}\right] dt + g \frac{\partial F}{\partial x} dW$$

So in our case ($f=\mu S, g=\sigma S$):

$$dC = \left[\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right] dt + \sigma S \frac{\partial C}{\partial S} dW$$

Merton's trick: consider a portfolio of value Π that eliminates risk

$$\Rightarrow \quad \underbrace{\Pi}_{\text{bonds}} = \alpha \underbrace{C}_{\text{option}} + \beta \underbrace{S}_{\text{stock}} \qquad \text{(replicating portfolio)}$$

$$d\Pi = \alpha\, dC + \beta\, dS$$

$$= \alpha \left[\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right] dt + \alpha \sigma S \frac{\partial C}{\partial S} dW + \beta \mu S dt + \beta \sigma S dW$$

to eliminate risk, we need $\beta = -\alpha \frac{\partial C}{\partial S}$

With that choice, we have

$$d\Pi = \alpha \left[\frac{\partial C}{\partial t} + \cancel{\mu S \frac{\partial C}{\partial S}} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - \cancel{\mu S \frac{\partial C}{\partial S}}\right] dt$$

Now there is no randomness in $d\Pi$ anymore, so Π has to grow with riskless rate r:

$$\Pi(t) = \Pi(0)e^{rt}, \text{ or } d\Pi = \Pi r\, dt = \alpha\left(C - \frac{\partial C}{\partial S} S\right) r\, dt$$

(Otherwise there would be the possibility of risk-free profit.)

Setting both expression for $d\Pi$ equal yields

$$\Rightarrow \boxed{\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r S \frac{\partial C}{\partial S} = r C}$$

This is the $\boxed{\text{Black-Scholes(-Merton) equation.}}$

Remarks:

- this is a partial differential equation (PDE), first order in time, second order in S
- we know the "initial condition" $C(S, t=T) =$ payoff, e.g. for European calls, we have $C(S,T) = \max(S-K, 0)$, $K =$ strike price, $T =$ expiration

  we want to solve for $C(S, t=0)$

  ↳ Black-Scholes eq. is a backward drift-diffusion equation

  ($\frac{\partial^2 C}{\partial S^2}$ is a "diffusion" term, $\frac{\partial C}{\partial S}$ a "drift" term)

- we have the boundary condition $C(S=0, t) = 0$ for all $t \in [0,T]$

- by a change of variables, the eq. can be transformed into

  $$\frac{\partial \theta}{\partial u} = \frac{1}{2} \frac{\partial^2 \theta}{\partial z^2} \quad (\theta = \theta(z,u)),$$ the heat equation

- option price $C(S,0)$ at time $t=0$ depends on the parameters r, σ, K, T, but not on μ. (Analogous to bin. tree model, where option price is independent of stock market probabilities.)

Hints for HW 6:

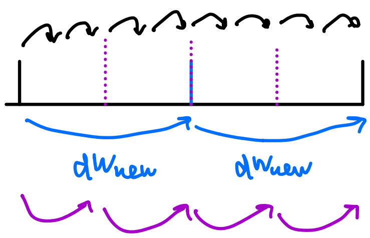
Problem 1: • S_N = Euler-Maruyama after N steps (corresponding to T, i.e., $\Delta t = \frac{T}{N}$)

• By def., Euler-Maruyama is inductive, i.e., it has to be implemented with a "for" loop. In the special case of GBM, one could also use cumprod.

• For strong/weak error: $\mathbb{E}$ is over ensemble, N is varied (# of steps in Euler-Maruyama)

• Note: weak error might be a bit hard to read off; try to fit a line by hand anyway.

• For the error rate, one could start with $N_{max} = 2**k$ $(k \simeq 10)$



$N = (2, 2^2, 2^3, \ldots, \underbrace{2**k}_{N_{max}})$

For each realization, create dW of length N_{max}

↳ with that, compute GBM

↳ compute $S_{N_{max}}$ (with the dW above)

↳ compute $S_{\frac{N_{max}}{2}}$ by using Euler-Maruyama with coarsened new

$dW_{new_{\frac{N_{max}}{2}}} = (dW_0 + dW_1, dW_1 + dW_2, \ldots)$

↳ repeat till $\frac{N_{max}}{2**k}$ ↖ or sth. smaller

Problem 4: Imagine two scenarios:

a) You want to keep the stock till after expiration. Is it then better to exercise early, e.g., when $S_t > K$, or at expiration?

b) You want to make profit immediately by exercising the option early, when $S_t > K$, and selling the stock. Is it better to exercise option, or to sell the option?