4.2 Connection between Black-Scholes Equation and Formula

B.-S. eq.: $\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$

European calls: "initial" condition $C(S,T) = \max(S-K,0)$

Generally: boundary condition $C(0,t) = 0$

One could do several changes of variables to reduce B.-S. eq. to the heat eq.

Ex.: One can remove the rC term by the following change of variables:

$$C(S,t) = B(S,\tau)e^{-r\tau}K \quad \text{with } \tau = T-t$$

then $\cdot\ \frac{\partial C}{\partial t} = \frac{\partial C}{\partial \tau}\underbrace{\frac{\partial \tau}{\partial t}}_{=-1} = -\left(\frac{\partial B}{\partial \tau}e^{-r\tau}K + \underbrace{B(-r)e^{-r\tau}K}_{-rC}\right)$

$$= -\frac{\partial B}{\partial \tau}e^{-r\tau}K + rC$$

$\cdot\ \frac{\partial C}{\partial S} = \frac{\partial B}{\partial S}e^{-r\tau}K$

$\Rightarrow$ B.-S. eq. becomes: $-\frac{\partial B}{\partial \tau}e^{-r\tau}K + \underline{rC} + rS\frac{\partial B}{\partial S}e^{-r\tau}K + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 B}{\partial S^2}e^{-r\tau}K = \underline{rC}$

$$\Rightarrow -\frac{\partial B}{\partial \tau} + rS\frac{\partial B}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2} = 0$$

with initial condition: $C(S,T) = B(S,0)K \overset{!}{=} \max(S-K,0)$

$$\Rightarrow \mathcal{B}(S,\tau=0) = \max\left(\frac{S}{K}-1, 0\right)$$

and boundary condition $\mathcal{B}(0,\tau) = 0$

With similar changes of variables one can remove the prefactors and $\frac{\partial C}{\partial S}$ term, and thus reduce the B.-S. eq. to the heat eq.:

$$\frac{\partial \theta}{\partial u} = \frac{1}{2}\frac{\partial^2 \theta}{\partial z^2} \quad \text{with initial condition } \theta(z,0) = \max\left(1-e^{-z}, 0\right)$$

$t \to \tau \to u$ $\qquad$ $S \to z$

(z as a fct. of S also depends on K)

the highest derivatives always remain

note: K is hidden in the change of variables from S to z.

The heat eq. can be solved with Fourier transform:

$$\hat{\theta}(p,u) = \frac{1}{\sqrt{2\pi}}\int e^{ipz}\,\theta(z,u)\,dz$$

$$\theta(z,u) = \frac{1}{\sqrt{2\pi}}\int e^{-ipz}\,\hat{\theta}(p,u)\,dp$$

keeping u fixed

$$\Rightarrow \text{heat eq. becomes} \quad \underbrace{\frac{1}{\sqrt{2\pi}}\int e^{-ipz}\frac{\partial\hat{\theta}(p,u)}{\partial u}\,dp}_{\frac{\partial\theta}{\partial u}} = \underbrace{\frac{1}{\sqrt{2\pi}}\int \frac{1}{2}(-ip)^2 e^{-ipz}\,\hat{\theta}(p,u)\,dp}_{\frac{1}{2}\frac{\partial^2\theta}{\partial z^2}}$$

$$\Rightarrow \text{need to solve} \quad \frac{\partial\hat{\theta}(p,u)}{\partial u} = \frac{1}{2}(-ip)^2\,\hat{\theta}(p,u) = -\frac{1}{2}p^2\,\hat{\theta}(p,u)$$

solution: $\hat{\theta}(p,u) = e^{-\frac{1}{2}p^2 u}\,\hat{\theta}(p,0)$

$$\Rightarrow \theta(z,u) = \frac{1}{\sqrt{2\pi}} \int e^{-ipz}\, \hat{\theta}(p,u)\, dp$$

$$= \frac{1}{\sqrt{2\pi}} \int e^{-ipz} e^{-\frac{1}{2}p^2 u}\, \underbrace{\hat{\theta}(p,0)}\, dp$$

$$= \frac{1}{\sqrt{2\pi}} \int e^{ipy}\, \theta(y,0)\, dy$$

$$= \frac{1}{2\pi} \int \underbrace{\left[\int e^{-ipz} e^{ipy} e^{-\frac{1}{2}p^2 u}\, dp \right]}_{\text{fct. of } y \text{ (and } z,u)} \theta(y,0)\, dy$$

Now: $$\int e^{-ip(z-y)} e^{-\frac{1}{2}up^2}\, dp = \int e^{-\frac{1}{2}up^2 - ip(z-y)}\, dp$$

$$= \int e^{-\frac{u}{2}\left[p^2 + 2p\frac{i(z-y)}{u} + \left(\frac{i(z-y)}{u}\right)^2 - \left(\frac{i(z-y)}{u}\right)^2\right]} dp$$

$$= \int e^{-\frac{u}{2}\left[p + \frac{i(z-y)}{u}\right]^2} dp\; e^{-\frac{u}{2}\left(-\left(\frac{i(z-y)}{u}\right)^2\right)}$$

$p + \frac{i(z-y)}{u} = \tilde{p}$

$\Rightarrow dp = d\tilde{p}$

$$= \int e^{-\frac{u}{2}\tilde{p}^2} d\tilde{p}\; e^{-\frac{(z-y)^2}{2u}}$$

$\tilde{p} = \frac{k}{\sqrt{u}}$

$$= \frac{1}{\sqrt{u}} \underbrace{\int e^{-\frac{k^2}{2}} dk}_{=\sqrt{2\pi}}\; e^{-\frac{(z-y)^2}{2u}}$$

$$= \sqrt{\frac{2\pi}{u}}\, e^{-\frac{(z-y)^2}{2u}}$$

convolution

$/\!/ \; (G_u * \theta(\cdot,0))(z), \quad G_u(x) = \frac{1}{\sqrt{2\pi u}} e^{-\frac{x^2}{2u}}$

$$\Rightarrow \boxed{\theta(z,u) = \frac{1}{\sqrt{2\pi u}} \int e^{-\frac{(z-y)^2}{2u}}\, \theta(y,0)\, dy}$$ is the solution to the heat eq. for ini. cond. $\theta(y,0)$

$\Rightarrow$ with $\Theta(y,0)=\max(1-e^{-y},0)$, we get

$$\Theta(z,u)=\frac{1}{\sqrt{2\pi u}}\int_0^\infty e^{-\frac{(z-y)^2}{2u}}(1-e^{-y})\,dy$$

Now substituting back all changes of variables would indeed lead to the Black-Scholes formula that we discussed before: (we omit the details here)

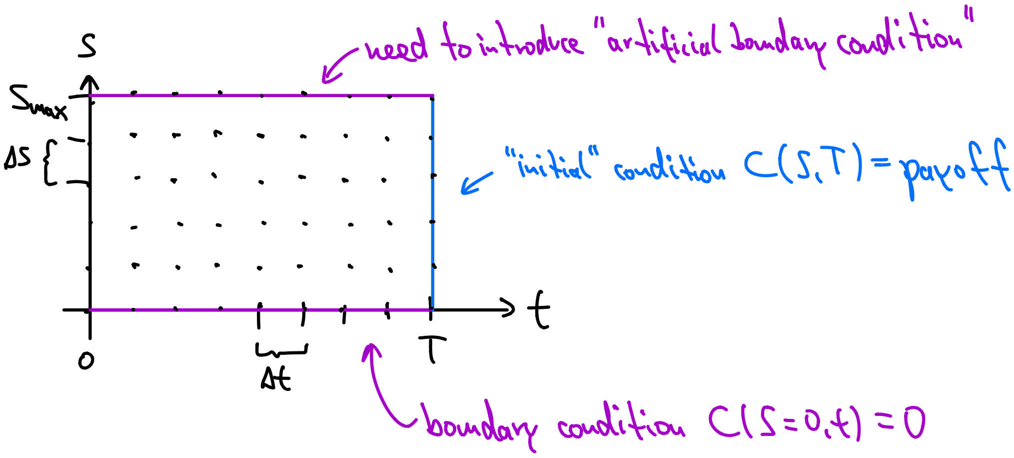
$$C(S,0)=S\,\Phi(x)-Ke^{-rT}\,\Phi(x-\sigma\sqrt{T})$$

with cumulative normal distribution $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy$

and $x=\dfrac{\ln(\frac{S}{K})+(r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$

4.3 Discrete Finite Differences

In order to solve PDEs such as the Black-Scholes eq., we need to discretize S, t:



What do we use for $C(S_{max}, t)$? For European calls, some possibilities are:

$\left(- C(S_{max}, t) = S_{max} - Ke^{-r(T-t)} \right.$, which is the interpolated behavior of C for $\left. S \to \infty \right)$

$\left(- C(S_{max}, t) = S_{max} - K \right)$

$- C(S_{max}, t) = S_{max}$, justified if $S_{max} >> K$

We have discretized $[0, S_{max}] \times [0, T]$ into a grid:

- M steps of size $\Delta t = \frac{T}{M}$, $t_j = j \cdot \Delta t$
- N steps of size $\Delta s = \frac{S_{max}}{N}$, $s_i = i \cdot \Delta s$

We abbreviate $C(s_i, t_j) = C_i^j$ $\leftarrow$ time (for j), $\leftarrow$ s (space) (for i)

Then:

$$\boxed{\frac{\partial C_i^j}{\partial t} = \frac{C_i^{j+1} - C_i^j}{\Delta t} + \underbrace{O(\Delta t)}_{\text{terms of order } \Delta t \text{, i.e. } \frac{O(\Delta t)}{\Delta t} \xrightarrow{\Delta t \to 0} \text{const}}}$$

$$\left(\text{Taylor: } \underbrace{C(s_i, t_j + \Delta t)}_{C_i^{j+1}} = \underbrace{C(s_i, t_j)}_{C_i^j} + \frac{\partial C_i^j}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 C_i^j}{\partial t^2} (\Delta t)^2 + O(\Delta t^3) \right)$$

For the s-derivative, we could choose

$\frac{\partial C_i^j}{\partial s} = \frac{C_{i+1}^j - C_i^j}{\Delta s} + O(\Delta s)$, but here one can do better

Taylor:

$$C(s_i+\Delta s) = C(s_i) + \frac{\partial C(s_i)}{\partial s}\Delta s + \frac{1}{2}\frac{\partial^2 C(s_i)}{\partial s^2}(\Delta s)^2 + \frac{1}{6}\frac{\partial^3 C_i}{\partial s^3}(\Delta s)^3 + O(\Delta s^4) \quad (1)$$

$$C(s_i-\Delta s) = C(s_i) - \frac{\partial C(s_i)}{\partial s}\Delta s + \frac{1}{2}\frac{\partial^2 C(s_i)}{\partial s^2}(\Delta s)^2 - \frac{1}{6}\frac{\partial^3 C_i}{\partial s^3}(\Delta s)^3 + O(\Delta s^4) \quad (2)$$

$$(1)-(2) \Rightarrow \quad C(s_i+\Delta s) - C(s_i-\Delta s) = 2\frac{\partial C(s_i)}{\partial s}\Delta s + O(\Delta s^3)$$

$\Rightarrow$ The centered derivative $\boxed{\frac{\partial C_i^j}{\partial s} = \frac{C_{i+1}^j - C_{i-1}^j}{2\Delta s} + O(\Delta s^2)}$ improves the error

Second derivative: $(1)+(2) \Rightarrow \boxed{\frac{\partial^2 C_i^j}{\partial s^2} = \frac{C_{i+1}^j - 2C_i^j + C_{i-1}^j}{(\Delta s)^2} + O(\Delta s^2)}$

Same error as centralized first derivative