Jacobs University
Spring 2020

February 10, 2020

Calculus and Linear Algebra II

Homework 1

Due on February 24, 2020

Problem 1 [**6 points**]

Take a look at the Riley, Hobson, Bence book *Mathematical Methods for Physics and Engineering* in Chapter 1.7.1 or any other suitable book and try to understand the method of proof by induction. Then use this method to prove the binomial theorem, i.e.,

$$(a+b)^N = \sum_{k=0}^{N} \binom{N}{k} a^{N-k} b^k.$$

Problem 2 [**4 points**]

Give a proof of the ratio test that was discussed in class. *Hint: Suppose* $|a_{k+1}/a_k|$ *converges to* $r < 1$*. Then, for large enough* N*,* $|a_{N+1}/a_N| < r$*. What can you conclude, using your knowledge about the geometric series?*

Problem 3 [**4 points**]

Suppose you would like to compute $\sum_{k=1}^{N} a_k$, and the summands are given as $a_k = b_k - b_{k-1}$ for some other sequence $(b_k)_{k\in\mathbb{N}}$. What is the value of $\sum_{k=1}^{N} a_k$? Use your answer to compute

(a)

$$\sum_{k=1}^{N} \Big(k^4 - (k-1)^4\Big),$$

(b)

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}.$$

Hint: Decompose the summands into partial fractions, i.e., find a, b *such that*

$$\frac{1}{k(k+1)} = \frac{a}{k} + \frac{b}{k+1}.$$

Problem 4 [4 points]
We define the power series

$$P(x) = \sum_{k=0}^{\infty} (k+1)x^k.$$

Compute the sum by using integration. What is $P(1/2)$?

Problem 5 [4 points]
One useful way to multiply two infinite series is given by the Cauchy product. Let us suppose we have two convergent series $\sum_{k=0}^{\infty} a_k = a$ and $\sum_{k=0}^{\infty} b_k = b$. A theorem says that if also $\sum_{k=0}^{\infty} |a_k|$ converges, then

$$\left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right) = \sum_{n=0}^{\infty}\sum_{j=0}^{n} a_{n-j}b_j.$$

Apply this formula to compute the product of $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ with itself. Did you expect the result?

Problem 6 [6 points]
As in class, we define

$$\mathbb{P}(n,k) := \binom{n}{k} p^k (1-p)^{n-k}$$

for some $p \in (0,1)$. Compute the variance of the binomial distribution, i.e., compute

$$\mathrm{Var} := \sum_{k=0}^{n} k^2 \mathbb{P}(n,k) - \left(\sum_{k=0}^{n} k\mathbb{P}(n,k)\right).$$

Problem 7 [2 points]
Consider the infinite series $\sum_{k=0}^{\infty} (-1)^k a_k$ with $a_k > 0$ for all $k \in \mathbb{N}$. Then the alternating series test (also called Leibniz test) says that if $a_{k+1} \le a_k$ and $\lim_{k\to\infty} a_k = 0$, then the series converges. Using this test, determine whether the following series converge:

(a) $\sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1}$,

(b) $\sum_{k=0}^{\infty} (-1)^{k+1} \frac{1}{\sqrt{k+5}}$.