Calculus and Linear Algebra II

Homework 1 (small corrections)

Due on February 24, 2020

Problem 1 [6 points]
Take a look at the Riley, Hobson, Bence book Mathematical Methods for Physics and Engineering in Chapter 1.7.1 or any other suitable book and try to understand the method of proof by induction. Then use this method to prove the binomial theorem, i.e.,

\[(a + b)^N = \sum_{k=0}^{N} \binom{N}{k} a^{N-k} b^k.\]

Problem 2 [4 points]
Give a proof of the ratio test that was discussed in class. Hint: Suppose \(|a_{k+1}/a_k|\) converges to \(r < 1\). Then, for large enough \(N\), \(|a_{N+1}/a_N| < R\) for some other \(r < R < 1\). What can you conclude, using your knowledge about the geometric series?

Problem 3 [4 points]
Suppose you would like to compute \(\sum_{k=1}^{N} a_k\), and the summands are given as \(a_k = b_k - b_{k-1}\) for some other sequence \((b_k)_{k \in \mathbb{N}}\). What is the value of \(\sum_{k=1}^{N} a_k\)? Use your answer to compute

(a) \(\sum_{k=1}^{N} \left(k^4 - (k - 1)^4\right)\),

(b) \(\sum_{k=1}^{\infty} \frac{1}{k(k+1)}\).

Hint: Decompose the summands into partial fractions, i.e., find \(a, b\) such that

\[\frac{1}{k(k+1)} = \frac{a}{k} + \frac{b}{k+1}.\]
Problem 4 [4 points]
We define the power series
\[
P(x) = \sum_{k=0}^{\infty} (k+1)x^k.
\]
Compute the sum by using integration. What is \(P(1/2)\)?

Problem 5 [4 points]
One useful way to multiply two infinite series is given by the Cauchy product. Let us suppose we have two convergent series \(\sum_{k=0}^{\infty} a_k = a\) and \(\sum_{k=0}^{\infty} b_k = b\). A theorem says that if also \(\sum_{k=0}^{\infty} |a_k|\) converges, then
\[
\left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} a_{n-j}b_j.
\]
Now consider \(f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\). Apply the formula above to compute \(f(x)f(y)\). Did you expect the result?

Problem 6 [6 points]
As in class, we define
\[
\mathbb{P}(n,k) := \binom{n}{k}p^k(1-p)^{n-k}
\]
for some \(p \in (0,1)\). Compute the variance of the binomial distribution, i.e., compute
\[
\text{Var} := \sum_{k=0}^{n} k^2\mathbb{P}(n,k) - \left(\sum_{k=0}^{n} k\mathbb{P}(n,k)\right)^2.
\]

Problem 7 [2 points]
Consider the infinite series \(\sum_{k=0}^{\infty} (-1)^k a_k\) with \(a_k > 0\) for all \(k \in \mathbb{N}\). Then the alternating series test (also called Leibniz test) says that if \(a_{k+1} \leq a_k\) and \(\lim_{k \to \infty} a_k = 0\), then the series converges. Using this test, determine whether the following series converge:
(a) \(\sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1}\),
(b) \(\sum_{k=0}^{\infty} (-1)^{k+1} \frac{1}{\sqrt{k+3}}\).