Jacobs University
Spring 2020 — February 24, 2020

Calculus and Linear Algebra II

Homework 2

Due on March 9, 2020

Problem 1 [**4 points**]

Suppose we are given two functions $f(x)$ and $g(x)$, and we would like to evaluate the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ for some $a \in \mathbb{R}$. Then we have that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)},$$

provided that numerator and denominator are not both equal to zero or infinity. If they are, one can use L'Hôpital's rule. If you do not know this rule, take a look at the Riley, Hobson, Bence book *Mathematical Methods for Physics and Engineering* in Chapter 4.7 (or any other suitable book). Using this rule, determine the limits

$$\lim_{x\to\infty} \frac{\ln(x)}{x^a} \quad \text{and} \quad \lim_{x\to\infty} \frac{e^x}{x^a},$$

for any $a > 0$. (Note: Keep the result of this exercise in mind: e^x grows faster than any polynomial, and $\ln(x)$ grows slower than any polynomial.)

Problem 2 [**7 points**]

In most applications, using a Taylor series "usually works", meaning that a few terms in the expansion give you a good approximation to the function, and the rest term is small enough. Here, we would like to look at a counter example to this. We consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{, for } x \neq 0 \\ 0 & \text{, for } x = 0. \end{cases}$$

(a) Visualize the function by drawing its graph. (Note: For such an exercise it is important to draw the qualitative features correctly, e.g., behavior at 0 and for large x, and possible maxima/minima.)

(b) Show that the function is continuous at 0.

(c) Compute the derivative $f'(x)$, and evaluate it at $x = 0$. Infer what the values of the higher derivatives $f^{(k)}(0)$ are.

(d) Based on your computations, what is the Talyor series of f around 0? Does the Taylor series of f around 0 converge to f?

Problem 3 [6 points]

Compute the Taylor series of $f(x) = \ln(1+x)$ around $x = 0$, for $-1 < x < 1$. Show that the rest term indeed converges to 0. Does the Taylor series also converge for $x = 1$? Does is converge for $x = -1$?

Problem 4 [2 points]

Consider the function $f(x, y) = e^{-x}y^3$, and the point $\vec{a} = (1, 1)$. At $\vec{a}$, in which direction does f increase the most? (If you like, visualize the function, e.g., with geogebra.org/3d, and check that your answer makes sense.)

Problem 5 [4 points]

Differentials are useful for approximating small changes in a function. Consider this example (taken from Folland's book). The volume of a right circular cone is given by $V(r, h) = \frac{1}{3}\pi r^2 h$, where r is the base radius and h the height. Consider explicitly $r = 3$ and $h = 5$.

(a) About how much does the volume increase if the height is increased to 5.02 and the radius to 3.01?

(b) If the height is increased to 5.02, by about how much should the radius be decreased to keep the volume constant?

Bonus Problem [7 points]

We consider the function

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0). \end{cases}$$

(a) Convince yourself that f is continuous at $(0, 0)$. (Choose a sequence (x_n, y_n) that converges to $(0, 0)$ and show that $f(x_n, y_n)$ converges to 0.)

(b) Compute the partial derivatives for $(x, y) \neq (0, 0)$. Then, using the definition of partial derivatives, compute $(\partial_x f)(0, 0)$ and $(\partial_y f)(0, 0)$.

(c) Show that f is not differentiable at $(0, 0)$ according to the definition of differentiability from class.