Jacobs University — March 16, 2020
Spring 2020

Calculus and Linear Algebra II

Homework 3

Due on March 30, 2020

Problem 1 [6 points]

In class, we discussed the Leibniz rule for computing the derivative of $\int_a^b f(x,t)\,\mathrm{d}t$. Generalize this rule to the case when the boundaries also depend on x, i.e., find the corresponding rule to compute the derivative of

$$I(x) = \int_{u(x)}^{v(x)} f(x,t)\,\mathrm{d}t.$$

Apply the new rule to compute the derivative of

$$G(x) = \int_{x}^{x^2} \frac{\sin(xt)}{t}\,\mathrm{d}t.$$

Problem 2 [6 points]

Use the method of Lagrange multipliers to find the smallest distance of the curve defined by $x^2 - 2\sqrt{3}xy - y^2 - 2 = 0$ to the origin $(0,0)$. Visualize the situation. *Hint: The distance is given by* $\sqrt{x^2+y^2}$*. But we could as well minimize the function* $f(x,y) = x^2 + y^2$ *(under the constraint above), which makes the computation a bit easier.*

Problem 3 [6 points]

Consider a curve in $\mathbb{R}^n$, which we represent by the continuously differentiable function $\vec{g}(t) : [a,b] \to \mathbb{R}^n$ with derivative $\vec{g}\,' \neq 0$. Then

$$\mathrm{d}\vec{g}(t) = \vec{g}\,'(t)\,\mathrm{d}t.$$

Thus, the length of an infinitesimal curve segment is given by

$$|\,\mathrm{d}\vec{g}(t)| = |\vec{g}\,'(t)|\,\mathrm{d}t = \sqrt{\left(\frac{\mathrm{d}g_1}{\mathrm{d}t}\right)^2 + \ldots + \left(\frac{\mathrm{d}g_n}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

Summing all these up gives us the arc length L of the curve,

$$L = \int_a^b |\vec{g}\,'(t)|\,\mathrm{d}t,$$

a formula that can also be rigorously proven when g is a C^1 function. Find the lengths of the following curves:

(a) $\vec{g}(t) = (M\cos(t), M\sin(t), Nt)$, $t \in [0, 2\pi]$,

(b) $\vec{g}(t) = (\frac{1}{3}t^3 - t, t^2)$, $t \in [0, 2]$.

But notice that we could as well reparametrize the curve. For any continuously differentiable bijective function $\varphi : [c, d] \to [a, b]$, the function $\vec{g}_\varphi(u) = \vec{g}(\varphi(u))$ would describe the same curve. Check that our formula indeed gives us the same curve length.

Problem 4 [6 points]

Without doing any computation, argue why there must be a minus sign in the following formula:

$$\operatorname{div}(\vec{F} \times \vec{G}) = \vec{G} \cdot (\operatorname{curl} \vec{F}) - \vec{F} \cdot (\operatorname{curl} \vec{G}).$$

Then, prove the formula by direct computation.

Problem 5 [6 points]

In three dimensions, Maxwell's equations for the electric field $\vec{E}$ and the magnetic field $\vec{B}$ in the vacuum read,

$$\operatorname{div} \vec{E} = 0, \ \operatorname{div} \vec{B} = 0, \ \operatorname{curl} \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \ \operatorname{curl} \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t},$$

with some constant $c > 0$.

(a) Show that all component of $\vec{E}$ and $\vec{B}$ satisfy the wave equation

$$\nabla^2 f = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}.$$

(b) When f is independent of time, the right-hand side of the wave equation vanishes. Show that $\nabla^2 \frac{1}{|\vec{x}|} = 0$ for $\vec{x} \neq 0$.