Jacobs University
Spring 2020

April 4, 2020

Calculus and Linear Algebra II

Homework 4

Due on April 20, 2020

Problem 1 [8 points]
Find the solution to the logistic growth model

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lambda y \left(1 - \frac{y}{k}\right)$$

by separation of variables, where $\lambda, k > 0$ are some parameters.

Problem 2 [4 points]
The point of this exercise is to give an example of an ODE where a solution exists but it is not unique. Show that this is the case for the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^{1/3},$$

for initial condition $y(0) = 0$. Specifically, show that there exist at least two solutions that satisfy the initial condition. Can you even give a whole family of solutions $y_\lambda(x)$, with each solution different for different $\lambda \in \mathbb{R}$?

Problem 3 [6 points]
We consider the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + \lambda y + 1,$$

for some parameter $\lambda \in \mathbb{R}$. Without solving the ODE explicitly discuss the qualitative behavior of the solutions for different values of the parameter λ and for different initial conditions. (I.e., what are the equilibrium positions, are they stable or not, how do the solutions behave for large x?)

Problem 4 [4 points]: The Curse of Dimension
Let us consider the standard basis $\vec{e}_1, \ldots, \vec{e}_n$ of $\mathbb{R}^n$.

(a) Compute the unit vector (= vector with length one) in the direction of the diagonal $\{\lambda(1, \ldots, 1) : \lambda \in \mathbb{R}\}$.

(b) Show that this unit vector is, in the limit $n \to \infty$, orthogonal to any basis vector $\vec{e}_j$, $j = 1, \ldots, n$.
Note: So with finite precision, they are actually orthogonal to each other for very large n, which contradicts our intuition from $\mathbb{R}^2$ or $\mathbb{R}^3$. This creates lots of problems for data mining, e.g., when performing a so-called Principal Component Analysis.

Problem 5 [8 points]: A least norm problem: Least-squares solutions of linear equations

(a) Generalize the method of Lagrange multipliers to the case where m constraints $G_j(\vec{x}) = 0$, $j = 1, \dots, m$ are present.

(b) Let us consider the following example of a so-called convex optimization problem (which are very relevant for machine learning). We aim at minimizing the function $f(\vec{x}) = |\vec{x}|^2 = \sum_{i=1}^{n} x_i^2$, subject to the constraint $\vec{G}(\vec{x}) = A\vec{x} - \vec{b} = \vec{0}$, where $\vec{x} \in \mathbb{R}^n$, $\vec{b} \in \mathbb{R}^m$, $\vec{G}(\vec{x}) \in \mathbb{R}^m$, and A a real $m \times n$ matrix with $\text{rank} A = m < n$. For example, A could come from some data set, and we want to find the solution to the system of linear equations with the smallest norm $|\vec{x}|^2$. Note that the matrix A has rank $m < n$, so it is not invertible. Hence, we cannot simply insert the constraint into f and compute the minimum. But luckily we know about Lagrange multipliers. So find the minimum of f under the m constraints $\vec{G}(\vec{x}) = \vec{0}$. *Hint: The matrix AA^T is invertible (why?), where A^T is the transpose matrix with entries $(A^T)_{ij} = A_{ji}$, i.e., where rows and columns are interchanged.*

Bonus Problem [8 points]

The equation of motion of a particle with charge $q > 0$ and mass $m > 0$ in an electric field $\vec{E}$ and a magnetic field $\vec{B}$ is

$$m \frac{\mathrm{d}^2 \vec{x}(t)}{\mathrm{d}t^2} = q \left(\left(\vec{x}'(t) \times \vec{B} \right) + \vec{E} \right),$$

where $\vec{x}(t) \in \mathbb{R}^3$ is the trajectory, and $\vec{x}'(t)$ the velocity of the particle. (This is the Lorentz force law.) Let us now consider the particular electric field

$$\vec{E} = (0, 0, E_3), \quad \text{with } E_3 > 0,$$

and magnetic field

$$\vec{B} = (B_1, 0, 0), \quad \text{with } B_1 > 0.$$

Find the solution to the equation of motion for initial data

$$\vec{x}(0) = \vec{0}, \quad \vec{x}'(0) = (0, v_0, 0), \quad \text{with } v_0 \geq 0.$$

Then, discuss the qualitative behavior of the solution depending on the parameter

$$\lambda = \frac{v_0}{E_3/B_1}.$$