Jacobs University
Spring 2020

April 20, 2020

Calculus and Linear Algebra II

Homework 5

Due on May 4, 2020

Problem 1 [**3 points**]
Consider the linear system of equations

$$\begin{aligned} 3x_1 + 2x_2 + 5x_3 &= 8, \\ x_1 + 2x_2 + 2x_3 &= 5, \\ 2x_1 + 2x_2 + 3x_3 &= 7. \end{aligned}$$

Use Cramer's rule to determine the solution x_3. (Just use Sarrus' rule to quickly compute the necessary determinants.)

Problem 2 [**4 points**]
Consider two vectors in $\mathbb{R}^3$,

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

In the standard basis

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

we can write the vectors as $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$ and $\vec{y} = y_1\vec{e}_1 + y_2\vec{e}_2 + y_3\vec{e}_3$. Now compute the determinant of the matrix

$$\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}.$$

(Note that our notation is a bit symbolic here, since we have put vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ as matrix entries; but that should not bother us.) The result should be familiar to you. Where have you encountered the resulting expression before?

Problem 3 [6 points]

Recall the definition of the classical adjoint $\mathrm{Adj}(A)$ of an $n \times n$ matrix A from class. We assume that A is an invertible matrix.

(a) Compute the determinant of $\mathrm{Adj}(A)$ in terms of the determinant of A.

(b) Show that the adjoint of the adjoint of A is guaranteed to equal A if $n = 2$, but not necessarily for $n > 2$.

Problem 4 [11 points]

In class, we discussed several properties of eigenvalues. Let us exemplify them for the general 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c, d \in \mathbb{R}$.

(a) Compute all the eigenvalues of A. Depending on a, b, c, d, how many real and complex eigenvalues are there?

(b) Find conditions on a, b, c, d such that A is invertible, and give an explicit formula for the inverse.

(c) Compute the sum of the eigenvalues. Is it indeed equal to $a + d$, i.e., the sum of the diagonal entries?

(d) Compute the product of the eigenvalues. Is it indeed equal to $\det(A)$?

(e) Now, plug the matrix A (instead of the number λ) into the characteristic polynomial, and verify that the Cayley-Hamilton theorem holds, i.e., that this gives zero.

Problem 5 [6 points]

Let us consider coordinate transformations in this exercises, which are indispensable for many-dimensional integration.

(a) First, let us consider two dimensions and the transformation from coordinates x, y to *polar coordinates* r, φ, where $\varphi \in [0, 2\pi)$, $r > 0$. The transformation is

$$x(r, \varphi) = r\cos(\varphi)$$
$$y(r, \varphi) = r\sin(\varphi).$$

Compute the Jacobian matrix of the function $\vec{f} : [0, \infty) \times [0, 2\pi) \to \mathbb{R}^2, \vec{f}(r, \varphi) = (r\cos(\varphi), r\sin(\varphi))$. Then, compute the determinant of the Jacobian matrix (which is usually called *the Jacobian*).

(b) **[Bonus]** In three dimensions, one of the most important coordinate transformations is the one to *spherical coordinates*. Here,

$$\begin{aligned} x(r,\varphi,\theta) &= r\cos(\varphi)\sin(\theta) \\ y(r,\varphi,\theta) &= r\sin(\varphi)\sin(\theta) \\ z(r,\varphi,\theta) &= r\cos(\theta), \end{aligned}$$

where $r > 0$, $\varphi \in [0, 2\pi)$, and $\theta \in [0, \pi)$. Compute also here the Jacobian matrix and then its determinant.

Why is this exercise so interesting? Recall the chain rule in one dimensional integration. When one changes coordinates from x to r in an integral $\int_a^b f(x)\,\mathrm{d}x$, then this changes the volume element from $\mathrm{d}x$ to $\frac{\mathrm{d}x(r)}{\mathrm{d}r}\,\mathrm{d}r$. Now, in several dimensions, the question is what replaces the derivative $\frac{\mathrm{d}x(r)}{\mathrm{d}r}$ when we want to generalize the chain rule. The right answer is that this is the determinant of the Jacobian matrix. This seems to be plausible, since we already know the interpretation of the determinant as a volume. It tells us locally how much volumes are stretched when we do a coordinate change. For example, in two-dimensions integrals of the form

$$\int_a^b \left(\int_c^d f(x,y)\,\mathrm{d}x \right) \mathrm{d}y = \int_a^b \int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

the volume element $\mathrm{d}x\,\mathrm{d}y$ transforms into $\det(J)\,\mathrm{d}r\,\mathrm{d}\varphi$, where $\det(J)$ is the determinant of the Jacobian matrix of the coordinate transformation from x, y to r, φ.
Here is an example for why this is so powerful. Consider the Gaussian integral

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x.$$

The Gaussian e^{-x^2} does not have a known antiderivative (it can not be expressed in terms of elementary functions). So there seems to be no hope of computing this (improper) integral. But we could write its square as

$$\begin{aligned} \left(\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x \right)^2 = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x \int_{-\infty}^{\infty} \mathrm{e}^{-y^2}\,\mathrm{d}y &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{e}^{-y^2}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-\left(x^2+y^2\right)}\,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

and now do a transformation to polar coordinates, i.e.,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-\left(x^2+y^2\right)}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\infty} \int_0^{2\pi} \mathrm{e}^{-\left(x(r,\varphi)^2+y(r,\varphi)^2\right)} \det(J)\,\mathrm{d}r\,\mathrm{d}\varphi,$$

where this notation means that r is integrated from 0 to ∞, and φ from 0 to 2π. This can now be computed, and thus we know the value of the original improper Gaussian integral. Can you find the result (bonus problem)?