Jacobs University
Spring 2020

February 12, 2020

Calculus and Linear Algebra II

Quiz 1

Instructions:

- Do all the work on this quiz paper.
- Show your work, i.e., write down the steps of your solution cleanly and readable.
- Electronic devices and notes are not allowed.

Name: Solutions

Problem 1 [5 points]

(a) Write down the formula for the binomial expansion:

$$(a+b)^n = \sum_{k=0}^{n} \boxed{\phantom{\binom{n}{k} a^{n-k} b^{k} \quad\quad\quad\quad\quad\quad}}$$

Solution:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

(b) Explicitly compute $\binom{10}{2}$.

Solution:

$$\binom{10}{2} := \frac{10!}{(10-2)!\,2!} = \frac{10 \cdot 9}{2} = 45.$$

Problem 1 (extra space)

Problem 2 [10 points]

(a) What is the value of the sum $\sum_{k=0}^{N} x^k$?

Solution:

This is the geometric series:

$$\sum_{k=0}^{N} x^k = \frac{1 - x^{N+1}}{1 - x}.$$

(b) For which x does $\sum_{k=0}^{\infty} x^k$ converge? For which x does it diverge?

Solution:

The series converges for $-1 < x < 1$, and diverges for $|x| \geq 1$.

(c) Apply the ratio test to $\sum_{k=1}^{\infty} \frac{k^3}{k!}$ in order to determine whether this series converges or diverges.

Solution:

Ratio test applied to $a_k = \frac{k^3}{k!}$:

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{(k+1)^3\, k!}{(k+1)!\, k^3} = \lim_{k\to\infty} \frac{(k+1)^2}{k^3} = 0 < 1.$$

So according to the ratio test, the series converges.

(d) Determine the radius of convergence of $\sum_{k=0}^{\infty} \frac{x^k}{k!}$

Solution:

According to the ratio test, using $a_k = \frac{1}{k!}$, the radius of convergence is

$$\rho = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| = \lim_{k\to\infty} \frac{(k+1)!}{k!} = \lim_{k\to\infty} (k+1) = \infty.$$

So the radius of convergence is infinite, meaning the power series converges for all $x \in \mathbb{R}$.

Problem 2 (extra space)