Jacobs University
Spring 2020

February 26, 2020

Calculus and Linear Algebra II

Quiz 2

Instructions:

- Do all the work on this quiz paper.
- Show your work, i.e., write down the steps of your solution cleanly and readable.
- Electronic devices and notes are not allowed.

Name: Solutions

Problem 1 [7 points]

(a) Let $f(x)$ be infinitely often differentiable at 0. Write down the definition of the (infinite) Taylor series of f around 0.

Solution:

The Taylor series f_T of f around 0 is

$$f_T(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

(b) Compute the Taylor series of $f(x) = \sin(x)e^{2x}$ up to quadratic order (i.e., including the x^2 term).

Solution:

We consider the Taylor series around 0. First, note that $f(0) = 0$. Then we compute the first and second derivatives and evaluate them at 0:

$$f'(x) = (\cos(x) + 2\sin(x))e^{2x}, \text{ so } f'(0) = 1,$$
$$f''(x) = (3\sin(x) + 4\cos(x))e^{2x}, \text{ so } f''(0) = 4.$$

Therefore,

$$f(x) = 0 + x + \frac{4}{2!}x^2 + R_2(x) = x + 2x^2 + R_2(x),$$

with some higher order remainder term $R_2(x)$.

(c) Does the integral $\int_{-1}^{1} x^{-2}dx$ exist as an improper integral (or a sum of improper integrals)? If yes, what is its value? If no, explain why.

Solution:

This is a bit tricky. A naive computation gives

$$\int_{-1}^{1} x^{-2}dx = -x^{-1}\Big|_{-1}^{1} = -1 - 1 = -2.$$

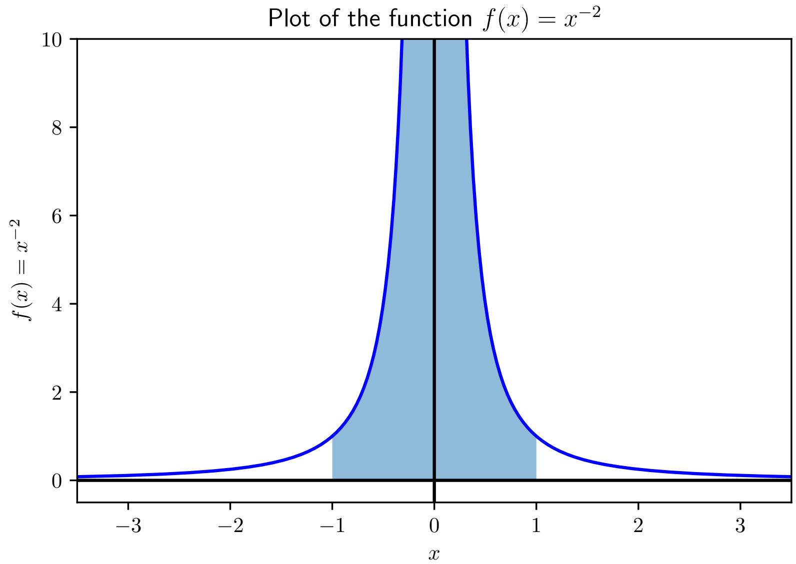
But this cannot be because the function x^{-2} is everywhere positive! Take a look at the picture on the next page.

The problem is that there is a singularity at $x = 0$. So by definition of improper integrals,

$$\int_{-1}^{1} x^{-2}dx = \int_{-1}^{0} x^{-2}dx + \int_{0}^{1} x^{-2}dx.$$

But the improper integrals on the right-hand side do not exist (they are infinite). So the whole integral does not exist.

Problem 1 (extra space)



Problem 2 [8 points]

(a) For $f(x,y) = x^2y^2 + \ln(x)\cos(y)$, compute the partial derivatives $\partial_x f$ and $\partial_y f$.

Solution:

$$\begin{aligned}\partial_x f &= 2xy^2 + \frac{1}{x}\cos(y),\\ \partial_y f &= 2x^2y - \ln(x)\sin(y).\end{aligned}$$

(b) Define what it means for a function $f : \mathbb{R}^n \to \mathbb{R}$ to be differentiable at $\vec{a} \in \mathbb{R}^n$.

Solution:

f is differentiable at $\vec{a}$ if there is some $\vec{m} \in \mathbb{R}^n$ such that

$$f(\vec{a} + \vec{h}) = f(\vec{a}) + \vec{m} \cdot \vec{h} + E(\vec{h}),$$

with $\frac{E(\vec{h})}{|\vec{h}|} \to 0$ as $\vec{h} \to 0$.

(c) Suppose we have three functions $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$, and $h : \mathbb{R}^n \to \mathbb{R}$ given. Simplify $\vec{\nabla}(fgh)$ with the product rule.

Solution:

We need to apply the product rule twice to find

$$\vec{\nabla}(fgh) = (\vec{\nabla} f)(gh) + f(\vec{\nabla}(gh)) = (\vec{\nabla} f)gh + f(\vec{\nabla} g)h + fg(\vec{\nabla} h).$$

(Note that the brackets here are usual brackets, and not arguments of functions.)

(d) What is the differential df of $f(x,y) = \sqrt{x^2+y^2}$?

Solution:

$$df = (\partial_x f)dx + (\partial_y f)dy = \frac{xdx + ydy}{\sqrt{x^2+y^2}}.$$

Problem 2 (extra space)