Calculus and Linear Algebra II

Quiz 3

Instructions:

- Do all the work on this quiz paper.

- Show your work, i.e., write down the steps of your solution cleanly and readable.

- Electronic devices and notes are not allowed.

Name: Solutions
Problem 1 [7 points]

Let \( f(x, y) = x \cos(y) - y^2 \).

(a) Compute the directional derivative of \( f \) at \((0, 0)\) in direction \((\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\).

\[ \text{Solution:} \]

We find

\[ (\nabla f)(x, y) = \begin{pmatrix} \cos(y) \\ -x \sin(y) - 2y \end{pmatrix} \Rightarrow (\nabla f)(0, 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]

Then the directional derivative at \((0, 0)\) in direction \(\vec{a} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\) is

\[ (D_{\vec{a}} f)(0, 0) = (\nabla f)(0, 0) \cdot \vec{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}. \]

(b) Now assume that \( x(t) = t^2 \) and \( y(t) = t^2 \). Compute \( \frac{d}{dt} f(x(t), y(t)) \).

\[ \text{Solution:} \]

We can use the chain rule:

\[ \frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \]

\[ = \cos(y)2t + (-x \sin(y) - 2y)2t \]

\[ = 2t \cos(t^2) - 2t^3 \sin(t^2) - 4t^3. \]

(c) Next, compute \( \frac{d}{dx} \int_0^{\pi/2} f(x, y)dy \).

\[ \text{Solution:} \]

We can use the Leibniz integral rule:

\[ \frac{d}{dx} \int_0^{\pi/2} f(x, y)dy = \int_0^{\pi/2} \frac{\partial f(x, y)}{\partial x} dy = \int_0^{\pi/2} \cos(y)dy = \sin(y) \bigg|_0^{\pi/2} = 1. \]
Problem 1 (extra space)
Problem 2 [8 points]

(a) Is the differential \( df = e^{-x+2y^2}(-dx + 4ydy) \) exact or inexact?

Solution:

If \( df = Adx + Bdy \), we need to check the condition \( \frac{\partial A}{\partial y} = \frac{\partial B}{\partial x} \). Here, we find

\[
\frac{\partial}{\partial y}(-e^{-x+2y^2}) = -4ye^{-x+2y^2}
\]

\[
\frac{\partial}{\partial x}(4ye^{-x+2y^2}) = -4ye^{-x+2y^2},
\]

so the differential is indeed exact. In fact, \( f = e^{-x+2y^2} \) has the differential \( df \) above.

(b) List what different types of critical points there are for a function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \).

Solution:

The critical points can be

- maxima (local or global)
- minima (local or global)
- saddle points
- degenerate points
- bonus point if you mention monkey saddle

(c) Find all critical points of \( f(x, y) = e^{x^2-y^2} \). What kind of critical points are they? (Try to answer by thinking about what this function looks like.)

Solution:

We have to check the condition \( \nabla f = 0 \). We find

\[
(\nabla f)(x, y) = \begin{pmatrix} 2x \\ -2y \end{pmatrix} e^{x^2-y^2}.
\]

This is zero only if \((x, y) = (0, 0)\), i.e., this is the only critical point. Clearly, the function has a minimum if we go along the \( x \)-axis, and a maximum if we go along the \( y \)-axis, so the critical point is a saddle point.

(d) [Only if you are already done with the other problems and are bored.] Find all critical points of \( f(x, y) = xy(12 - 3x - 4y) \).

Solution:

Checking the condition \( \nabla f = 0 \) yield the four critical points \((0, 0), (4, 0), (0, 3), (\frac{4}{3}, 1)\).
Problem 2 (extra space)