Jacobs University
Spring 2020

March 25, 2020

Calculus and Linear Algebra II

Quiz 4

Instructions:

- Do all the work on this quiz paper.
- Show your work, i.e., write down the steps of your solution cleanly and readable.
- Electronic devices and notes are not allowed.

Name: Solutions

Problem 1 [8 points]

Let $f(x,y) = 1 + xy$. Using the method of Lagrange multipliers, find the critical points of f on the unit circle, i.e., under the constraint $G(x,y) = x^2 + y^2 - 1 = 0$. (You do not need to check whether they are maxima or minima or neither, but you can try to find out if time is left.)

Solution:

We introduce the Lagrange multiplier λ and the Lagrange function

$$L(x,y,\lambda) = f(x,y) - \lambda G(x,y) = 1 + xy - \lambda(x^2 + y^2 - 1)$$

Then we need to solve a system of three equations:

$$\begin{aligned} \partial_x L &= y - 2x\lambda = 0 \\ \partial_y L &= x - 2y\lambda = 0 \\ \partial_\lambda L &= x^2 + y^2 - 1 = 0. \end{aligned}$$

From the first equation we get $\lambda = y/(2x)$. Plugging this into the second equation gives $x^2 = y^2$. Plugging this into the third equation gives $x^2 = 1/2$, i.e., $x = \pm 1/\sqrt{2}$. Accordingly, $y = \pm 1/\sqrt{2}$. Thus, there are four critical points:

$$\Big(1/\sqrt{2}, 1/\sqrt{2}\Big),\ \Big(-1/\sqrt{2}, -1/\sqrt{2}\Big),\ \Big(1/\sqrt{2}, -1/\sqrt{2}\Big),\ \Big(-1/\sqrt{2}, 1/\sqrt{2}\Big).$$

Plugging these into the function f, we find that the first two are maxima (here $f = 3/2$) and the last two are minima. If you like, visualize the function and the constraint with geogebra.

Problem 1 (extra space)

Problem 2 [7 points]

(a) Let

$$\vec{f}:\mathbb{R}^2\to\mathbb{R}^3,(x,y)\mapsto \vec{f}(x,y)=\begin{pmatrix} x^2y \\ \sin(x) \\ e^{x+y}\end{pmatrix}.$$

Determine the Jacobian matrix of $\vec{f}$.

Solution:

Just recall the definition from class:

$$J_{\vec{f}}(x,y)=\begin{pmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \\ \partial_x f_3 & \partial_y f_3 \end{pmatrix}=\begin{pmatrix} 2xy & x^2 \\ \cos(x) & 0 \\ e^{x+y} & e^{x+y}\end{pmatrix}.$$

(b) Let

$$\vec{f}:\mathbb{R}^3\to\mathbb{R}^3,(x,y,z)\mapsto \vec{f}(x,y,z)=\begin{pmatrix} x^2yz \\ x^2+y^2+z^2 \\ x^2+y^2+z^2\end{pmatrix}.$$

Determine $\text{curl}\vec{f}$. Then, compute $\text{div}\,(\text{curl}\vec{f}\,)$.

Solution:

We find

$$\text{curl}\vec{f}=\begin{pmatrix}\partial_x\\ \partial_y \\ \partial_z\end{pmatrix}\times\begin{pmatrix} f_1 \\ f_2 \\ f_3\end{pmatrix}=\begin{pmatrix}\partial_y f_3-\partial_z f_2 \\ \partial_z f_1-\partial_x f_3 \\ \partial_x f_2-\partial_y f_1\end{pmatrix}=\begin{pmatrix} 2y-2z \\ x^2y-2x \\ 2x-x^2z\end{pmatrix}.$$

Then,

$$\text{div}\,(\text{curl}\vec{f}=\begin{pmatrix}\partial_x\\ \partial_y \\ \partial_z\end{pmatrix}\cdot\begin{pmatrix} 2y-2z \\ x^2y-2x \\ 2x-x^2z\end{pmatrix}=0+x^2-x^2=0.$$

This result was expected, since we showed in class that $\text{div}\,(\text{curl}\vec{f}=0$ always holds (when $\vec{f}$ is C^2).

Problem 2 (extra space)