Calculus and Linear Algebra II

Quiz 4

Instructions:

• Do all the work on this quiz paper.

• Show your work, i.e., write down the steps of your solution cleanly and readable.

• Electronic devices and notes are not allowed.

Name: Solutions
Problem 1 [8 points]

Let \( f(x, y) = 1 + xy \). Using the method of Lagrange multipliers, find the critical points of \( f \) on the unit circle, i.e., under the constraint \( G(x, y) = x^2 + y^2 - 1 = 0 \). (You do not need to check whether they are maxima or minima or neither, but you can try to find out if time is left.)

Solution:

We introduce the Lagrange multiplier \( \lambda \) and the Lagrange function

\[
L(x, y, \lambda) = f(x, y) - \lambda G(x, y) = 1 + xy - \lambda(x^2 + y^2 - 1)
\]

Then we need to solve a system of three equations:

\[
\begin{align*}
\partial_x L &= y - 2x\lambda = 0 \\
\partial_y L &= x - 2y\lambda = 0 \\
\partial_\lambda L &= x^2 + y^2 - 1 = 0.
\end{align*}
\]

From the first equation we get \( \lambda = y/(2x) \). Plugging this into the second equation gives \( x^2 = y^2 \). Plugging this into the third equation gives \( x^2 = 1/2 \), i.e., \( x = \pm 1/\sqrt{2} \). Accordingly, \( y = \pm 1/\sqrt{2} \). Thus, there are four critical points:

\[
\left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \quad \left( -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right), \quad \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right), \quad \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right).
\]

Plugging these into the function \( f \), we find that the first two are maxima (here \( f = 3/2 \)) and the last two are minima. If you like, visualize the function and the constraint with geogebra.
Problem 1 (extra space)
Problem 2 [7 points]

(a) Let \( \vec{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^3, (x, y) \mapsto \vec{f}(x, y) = \begin{pmatrix} x^2 y \\ \sin(x) \\ e^{x+y} \end{pmatrix} \).

Determine the Jacobian matrix of \( \vec{f} \).

Solution:

Just recall the definition from class:

\[
J_{\vec{f}}(x, y) = \begin{pmatrix}
\frac{\partial}{\partial x} f_1 & \frac{\partial}{\partial y} f_1 \\
\frac{\partial}{\partial x} f_2 & \frac{\partial}{\partial y} f_2 \\
\frac{\partial}{\partial x} f_3 & \frac{\partial}{\partial y} f_3
\end{pmatrix} = \begin{pmatrix} 2xy & x^2 \\
\cos(x) & 0 \\
e^{x+y} & e^{x+y} \end{pmatrix}.
\]

(b) Let

\( \vec{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3, (x, y, z) \mapsto \vec{f}(x, y, z) = \begin{pmatrix} x^2 y z \\
x^2 + y^2 + z^2 \\
x^2 + y^2 + z^2 \end{pmatrix} \).

Determine \( \text{curl} \vec{f} \). Then, compute \( \text{div} (\text{curl} \vec{f}) \).

Solution:

We find

\[
\text{curl} \vec{f} = \begin{pmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{pmatrix} \times \begin{pmatrix} f_1 \\
f_2 \\
f_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y} f_3 - \frac{\partial}{\partial z} f_2 \\
\frac{\partial}{\partial z} f_1 - \frac{\partial}{\partial x} f_3 \\
\frac{\partial}{\partial x} f_2 - \frac{\partial}{\partial y} f_1
\end{pmatrix} = \begin{pmatrix} 2y - 2z \\
x^2 y - 2x \\
x^2 + 2x - x^2 z
\end{pmatrix}.
\]

Then,

\[
\text{div} (\text{curl} \vec{f}) = \begin{pmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{pmatrix} \cdot \begin{pmatrix} 2y - 2z \\
x^2 y - 2x \\
x^2 + 2x - x^2 z
\end{pmatrix} = 0 + x^2 - x^2 = 0.
\]

This result was expected, since we showed in class that \( \text{div} (\text{curl} \vec{f}) = 0 \) always holds (when \( \vec{f} \) is \( C^2 \)).
Problem 2 (extra space)