Jacobs University
Spring 2020

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Calculus and Linear Algebra II

Quiz 5

Instructions:

- Do all the work on this quiz paper.
- Show your work, i.e., write down the steps of your solution cleanly and readable.
- Electronic devices and notes are not allowed.

Name: Solutions

Problem 1 [8 points]

Find the solution to the first-order ordinary differential equation $\frac{dy}{dx} = \lambda\sqrt{y}$ for any $\lambda > 0$ using separation of variables (it is ok to only find $y(x)$ for $x > 0$). Write down your solution using the general initial data $y(0) = y_0 \geq 0$.

Solution:

We use separation of variables:

$$\frac{dy}{\sqrt{y}} = \lambda dx \quad \Rightarrow \quad \int \frac{dy}{\sqrt{y}} = \int \lambda dx \quad \Rightarrow \quad 2\sqrt{y} = \lambda x + C.$$

So the general solution is

$$y(x) = \left(\frac{\lambda}{2}x + \frac{1}{2}C\right)^2.$$

As a side note: Note that the constant C is determined by the initial conditions, so it does not matter if we use $\frac{1}{2}C$ or just replace this term by C. (We could have introduced another letter here to distinguish the two constants, but we don't bother.) So we could also just write

$$y(x) = \left(\frac{\lambda}{2}x + C\right)^2.$$

Just to be sure, check that this function really satisfies the differential equation by computing the derivative.

Now we are given the value of the function at $x = 0$. We compute

$$y(0) = C^2.$$

This should be equal to y_0, so $C = \sqrt{y_0}$ and we can write our solution as

$$y(x) = \left(\frac{\lambda}{2}x + \sqrt{y_0}\right)^2.$$

Problem 1 (extra space)

Problem 2 [7 points]
Compute the determinant of

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix}.$$

Clearly state which property or method you use in each step of the computation.

Solution:
There are several ways to do this. Let us first compute it by bringing it into upper triangular form:

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{pmatrix} = 1 \times 1 \times (-3) = -3.$$

In the first step, we added (-1) times the first row to the third one, and in the second step we added (-1) times the second row to the third one. Then the determinant is the product of the diagonal values.

Alternatively, we could use a Laplace expansion. There is a zero in the first column, so let us choose an expansion using this first column. We find

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix} &= (-1)^{1+1} \det \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} + 0 + (-1)^{1+3} \det \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \\ &= 1 \times 2 - 2 \times 3 + 2 \times 2 - 3 \times 1 \\ &= -3. \end{aligned}$$

Problem 2 (extra space)