Calculus and Linear Algebra II

Prof. Sören Petrat
Research area: Mathematical Physics

Organization:
- basic information: campusnet
- up-to-date info: class website
- class: Mon/Wed, 14:15-15:30 (Conrad-Nabor Lecture Hall)
- if you have questions: ask me after class (won’t be able to answer all emails)
- if you need prerequisite waiver: after class

- grade only based on final exam
- books: see website
- lecture notes: see website
- TAs: Dzmitry, Abhik, Cyrine, Miruna

- tutorials: questions, individual help with exercises, practice
  - 3 time slots, tbd, starting next week
  - really important!

- exercises: homework sheets (see website, moodle)
  - hand in after class on due date
  - or put in mailbox at entrance of Research I
  - moodle exercises
  - biweekly quizzes
- from each pool of exercises, 2 will be on final exam!
1. Some Extra Part I Topics

1.1 Binomial Expansion

We would like to expand \((a + b)^n\)

Example: \((a + b)^0 = 1\)
\((a + b)^1 = a + b\)
\((a + b)^2 = a^2 + 2ab + b^2\)
\((a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\)

= # of possibilities to distribute 2 a's and 1 b into 3 slots: aab, aba, ba

Generally: \((a + b)^{n+1} = \binom{n}{k} a^{n-k} b^k + \ldots + b^{n+1}\)

\[(a + b)^{n+1} = \left(\sum \binom{n}{k} a^{n-k} b^k + \ldots + b^{n+1}\right) (a + b)\]

General pattern: 1

\[\begin{array}{c|c|c|c|c|c|c|c|c}
\hline
k & 0 & 1 & 2 & 3 & 4 & 5 & \ldots \\
\hline
\binom{n}{k} & \ 1 & \ 1 & \ 2 & \ 1 & \ 4 & \ 6 & \ 4 & \ 1 \\
\hline
\end{array}\]
Formula: \( C(n,k) = \binom{n}{k} := \frac{n!}{(n-k)!k!} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{1\cdot2\cdot3\ldots k} \)

"\( n \) choose \( k \" (sometimes: \( C(n,k) = \binom{n}{k} \))

Theorem: \((a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \ldots + \binom{n}{n} b^n \)

\( \sum \) from \( k=0 \) to \( n \) Binomial expansion

Note: \( 0! := 1 \) so \( \binom{n}{0} = \frac{n!}{0!0!} = 1 \) \( \binom{n}{1} = \frac{n!}{(n-1)!1!} = 1 \)

- Symmetry: \( \binom{n}{k} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k} \)
- Property from above:

\[ \binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(n-k+1)!(k-1)!} \cdot k + \frac{n!}{(n-k)!(k-1)!} \cdot (n-k+1) \]

\[ = \frac{n!k}{(n-k+1)!k!} + \frac{n!(n-k+1)}{(n-k)!k!} \]

\[ = \frac{n!(k+n-k+1)}{(n-k+1)!k!} = \frac{(n+1)!}{(n+1-k)!k!} = \binom{n+1}{k} \checkmark \)

Different perspective: Galton board

So-so chance to go left or right (as in coin toss)

\( \leftarrow \) Gauss distribution!

(\( k \) in a continuous limit)
how many paths end up in box $k$?

(n-k) times left, k times right => \( \binom{n}{k} \) paths

\[ \Rightarrow \text{probability for landing in box } k: \quad \binom{n}{k} \ p^k \ (1-p)^{n-k} = \text{TP}(n,k) \]

\[ \text{spe}(0,1) = \text{probability to go right} \]

(before we had $p = \frac{1}{2}$)

\[ \Rightarrow \text{Probability to end up somewhere:} \quad \sum_{k=0}^{n} \text{TP}(n,k) = \sum_{k=0}^{n} \binom{n}{k} \ p^k \ (1-p)^{n-k} = (p + (1-p))^n = 1 \checkmark \]

Question: What is expectation value $\mathbb{E} = \sum_{k=0}^{n} k \text{TP}(n,k)$?

\[ \mathbb{E} = \sum_{k=0}^{n} k \binom{n}{k} \ p^k \ (1-p)^{n-k} = (1-p)^n \sum_{k=0}^{n} \binom{n}{k} \left( \frac{p}{1-p} \right)^k \]

\[ \text{need to compute} \quad \sum_{k=0}^{n} \binom{n}{k} x^k \]

\[ \text{trick:} \quad x \cdot \frac{d}{dx} \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k \]

\[ = \sum_{k=0}^{n} \binom{n}{k} x^{k-1} \]

\[ = (x+1)^n \]

\[ \Rightarrow x \cdot \frac{d}{dx} (x+1)^n = x \cdot n (x+1)^{n-1} \]

\[ \Rightarrow \mathbb{E} = (1-p)^n \cdot x \cdot n (x+1)^{n-1} = \frac{(1-p)^n (\frac{p}{1-p})}{(1-p)^n} \cdot n \frac{p}{1-p} = n \cdot p \]

\[ \text{(1-p)^{n-1} p} \]

\[ = \left( \frac{p}{1-p} + \frac{1-p}{1-p} \right)^n = \left( \frac{1}{1-p} \right)^n \]

\[ \text{Similarly: variance} \quad \sum_{k=0}^{n} k^2 \text{TP}(n,k) - \mathbb{E}^2 \quad \text{(see HW)} \]