Session 2
Feb. 5, 2020

1.2 Infinite Series

Let us consider $a_0 + a_1 + a_2 + \dots + a_N = \sum_{k=0}^{N} a_k =: S_N$, called partial sum

Ex.: • arithmetic series: $\sum_{k=0}^{N} k$

e.g., $N=6$: $\sum_{k=0}^{6} k = 0 + 1 + 2 + 3 + 4 + 5 + 6 = 7 \cdot \frac{6}{2} = 21$

general: $\sum_{k=0}^{N} k = 0 + 1 + 2 + \dots + (N-1) + N = (N+1)\,\frac{N}{2}$

(The story is that Gauß figured this out in elementary school.)

• geometric series: $\sum_{k=0}^{N} x^k = ?$

we compute: $\sum_{k=0}^{N} x^k - x\sum_{k=0}^{N} x^k = \underbrace{\sum_{k=0}^{N} x^k}_{= 1 + x + x^2 + \dots + x^N} - \underbrace{\sum_{k=0}^{N} x^{k+1}}_{= x + x^2 + \dots + x^N + x^{N+1}} = 1 - x^{N+1}$

$$\sum_{k=0}^{N} x^k - x\sum_{k=0}^{N} x^k = (1-x)\sum_{k=0}^{N} x^k$$

$$\Rightarrow \boxed{\sum_{k=0}^{N} x^k = \frac{1-x^{N+1}}{1-x}}$$

back to $S_N = \sum_{k=0}^{N} a_k$

Observation: $(S_N)_{N\in\mathbb{N}}$ is a sequence

$\Rightarrow$ it is either $\longrightarrow$ convergent, i.e., $\lim_{N\to\infty} S_N =: \sum_{k=0}^{\infty} a_k$ exists

$\searrow$ or divergent

Ex.: • $\sum_{k=0}^{\infty} x^k = \lim_{N\to\infty} \frac{1-x^N}{1-x} = \frac{1-\lim_{N\to\infty} x^N}{1-x} = \begin{cases} \text{convergent to } \frac{1}{1-x} \text{ for } -1<x<1 \\ \text{divergent to } +\infty \text{ for } x\geq 1 \\ \text{divergent for } x\leq -1 \end{cases}$

• $\sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{1+\frac{1}{2}} = \frac{1}{3/2} = \frac{2}{3}$

There are several criteria to determine whether $\sum_{k=0}^{\infty} a_k$ is convergent or not:

• necessary condition: $\lim_{k\to\infty} a_k = 0$

Ex.: $\sum_{k=0}^{\infty} k^{\frac{3}{2}}$ or $\sum_{k=0}^{\infty} \frac{k}{k+1}$ are such divergent

• $\boxed{\text{comparison test:}}$ let $0 \leq a_k \leq b_k \quad \forall k \in \mathbb{N}$

↳ If $\sum_{k=0}^{\infty} b_k$ converges, then so does $\sum_{k=0}^{\infty} a_k$

↳ If $\sum_{k=0}^{\infty} a_k$ diverges, then so does $\sum_{k=0}^{\infty} b_k$

Ex.: $b_k = \frac{1}{k+1}$, i.e., $\sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} \frac{1}{k+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$

↳ compare with $\sum_{k=0}^{\infty} a_k = 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{=\frac{1}{2}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{=\frac{1}{2}} + \dots$

$\Downarrow$

diverges

thus also $\sum_{k=0}^{\infty} \frac{1}{k+1}$ diverges

- ratio test: If $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$ is $\begin{cases} < 1, \text{ then series converges} \\ > 1 \text{ or } \infty, \text{ then series diverges} \\ = 1 \text{ or doesn't exist, then test is inconclusive} \end{cases}$

(reason/proof: see HW)

Ex.: • $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ for $x \in \mathbb{R}$

$$\lim_{k\to\infty} \left| \frac{x^{k+1}/(k+1)!}{x^k/k!} \right| = \lim_{k\to\infty} \left| \frac{x^{k+1}}{x^k} \frac{k!}{(k+1)!} \right| = \lim_{k\to\infty} \frac{|x|}{k+1} = 0 \quad \forall x \in \mathbb{R}$$

$\Rightarrow$ series converges

• $\sum_{k=0}^{\infty} \frac{1}{k+1}$ $\to$ we find $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{1/k+2}{1/k+1} = \lim_{k\to\infty} \frac{k+1}{k+2} = 1$

$\Rightarrow$ test inconclusive

• $(a_k)_{k\in\mathbb{N}} = (a_0, 0, a_2, 0, a_4, 0, a_6, 0, \dots)$

$\Rightarrow$ test not applicable (inconclusive)

1.3 Power Series

Definition: $\sum_{k=0}^{\infty} a_k x^k$ is called power series.

$\varrho := \sup \{ |x| : \sum_{k=0}^{\infty} a_k x^k \text{ converges} \}$ is called radius of convergence.

(Recall: let $A \subset \mathbb{R}$, then • supremum $\sup A :=$ smallest upper bound of A

• infimum $\inf A :=$ largest lower bound of A

Ex.: $\sup(-\frac{1}{2}, 5) = 5$, $\inf(-\frac{1}{2}, 5) = -\frac{1}{2}$)

So by definition, $\sum_{k=0}^{\infty} a_k x^k$ converges if $-\rho < x < \rho$.

ratio test: convergence if $\lim_{k\to\infty} \left| \frac{a_{k+1} x^{k+1}}{a_k x^k} \right| = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| |x| < 1$

$\Rightarrow$ need $|x| < \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right|$ $\Rightarrow$ $\boxed{\rho = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| \text{ if this limit exists}}$

<u>Ex.:</u> • $\sum_{k=0}^{\infty} (2x)^k = \sum_{k=0}^{\infty} \underbrace{2^k}_{=a_k} x^k$

we find $\rho = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| = \lim_{k\to\infty} \frac{2^k}{2^{k+1}} = \frac{1}{2}$

$\Rightarrow$ series converges for $-\frac{1}{2} < x < \frac{1}{2}$ (but note: does not converge <u>at</u> $x = \frac{1}{2}$)

• above we found that $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges $\forall x \in \mathbb{R}$, i.e., $\rho = \infty$

Note: ρ can be 0, some number > 0, or ∞

Let us assume $f(x) = \sum_{k=0}^{\infty} a_k x^k$ has convergence radius $\rho = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right|$

What about $\boxed{f'(x) = \sum_{k=1}^{\infty} a_k \cdot k \, x^{k-1}}$?

$$\Rightarrow \tilde{\rho} = \lim_{k\to\infty} \left| \frac{a_k \cdot k}{a_{k+1}(k+1)} \right| = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| \cdot \underbrace{\lim_{k\to\infty} \frac{k}{k+1}}_{=1} = \rho$$

$\Rightarrow$ a power series and its derivative have the same radius of convergence

Similarly: $\boxed{\int \sum_{k=0}^{\infty} a_k x^k \, dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}}$

<u>Ex.:</u> $\frac{d}{dx} \underbrace{\sum_{k=0}^{\infty} \frac{x^k}{k!}}_{f(x)} = \sum_{k=1}^{\infty} \frac{k}{k!} x^{k-1} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = 1 + x + \frac{x^2}{2} + \ldots = \underbrace{\sum_{k=0}^{\infty} \frac{x^k}{k!}}_{f(x)}$

$\Rightarrow \sum_{k=0}^{\infty} \frac{x^k}{k!}$ is its own derivative!

$\Rightarrow \boxed{\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x = \exp(x)}$, exponential function!