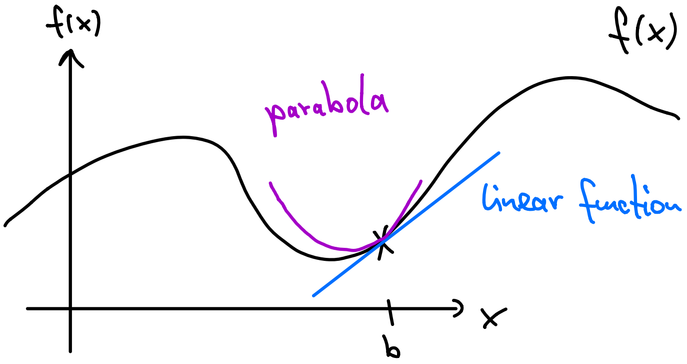
## 1.4 Taylor Series

Session 3
Feb. 10, 2020




Goal: approximate $f(x)$ near $b$ by a power series

we compute: $\int_b^x f'(y)\,dy = f(x) - f(b)$

$$\Rightarrow f(x) = f(b) + \int_b^x f'(y)\,dy$$

integration by parts: $\int_b^x f'(y) \cdot \underbrace{1}_{=\frac{d}{dy}(y-x)} dy = (y-x) f'(y) \Big|_b^x - \int_b^x (y-x) f''(y)\,dy$

(recall: $h \cdot g = \int (hg)' = \int h'g + \int hg'$
$\Rightarrow \int hg' = hg - \int h'g$)

$$= (x-b) f'(b) + \int_b^x (x-y) f''(y)\,dy$$

$$\Rightarrow f(x) = \underbrace{f(b) + (x-b) f'(b)}_{\text{linear approximation}} + \int_b^x (x-y) f''(y)\,dy$$

another integration by parts yields (check the computation!):

$$f(x) = \underbrace{f(b) + (x-b) f'(b) + \frac{(x-b)^2}{2} f''(b)}_{\text{second order approximation (parabola)}} + \int_b^x \frac{(x-y)^2}{2} f'''(y)\,dy$$

$\boxed{\begin{array}{l}\underline{\text{Theorem (Taylor expansion):}}\\[1em] \text{Let } f:\mathbb{R}\to\mathbb{R} \text{ be } (N+1) \text{ times continuously differentiable on } [b,x]. \text{ Then}\\[1em] \displaystyle f(x)=\sum_{k=0}^{N}\frac{(x-b)^k}{k!}f^{(k)}(b)+\underbrace{\int_b^x\frac{(x-y)^N}{N!}f^{(N+1)}(y)\,dy}_{=:R_N(x)\text{, called remainder}}\end{array}}$

Remarks:

- $\exists$ other formulas for the remainder, e.g., the Lagrange form:

  $\boxed{R_N(x)=\frac{f^{(N+1)}(t)}{(N+1)!}(x-b)^{N+1}}$ for some $t\in[b,x]$ (follows from mean-value theorem)

- $\sum_{k=0}^{\infty}\frac{(x-b)^k}{k!}f^{(k)}(b)$ is called Taylor series of $f$ around $b$

  If $b=0$, it is called Maclaurin series

- If $f$ is arbitrarily often differentiable <u>and</u> $R_N(x)\xrightarrow{N\to\infty}0$ for some $x$ (near $b$),

  then $f(x)=\sum_{k=0}^{\infty}\frac{(x-b)^k}{k!}f^{(k)}(b)$

- Note: The Taylor series for $\infty$ often differentiable $f$ might
  - converge to $f$ for some or all $x$
  - converge, but not to $f$ (if $R_N(x)$ does not converge to $0$)
  - diverge (except at $x=b$)

Examples:

- $f(x) = e^x \Rightarrow f^{(k)}(x) = e^x \Rightarrow f^{(k)}(0) = e^0 = 1$

  $\Rightarrow$ Taylor series of $f$ around $b=0$ is $\sum_{k=0}^{\infty} \frac{x^k}{k!}$

  We already know that indeed $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for all $x \in \mathbb{R}$

- $f(x) = \sin x \Rightarrow f'(x) = \cos x$

$$f''(x) = -\sin x$$
$$f'''(x) = -\cos x$$
$$f^{(4)}(x) = \sin x$$
$$\vdots$$

Let us choose $b=0$: since $\sin(0)=0$, $\cos(0)=1$, we have $f^{(k)}(0) = \begin{cases} 0 & , k \text{ even} \\ (-1)^{\frac{k-1}{2}} & , k \text{ odd} \end{cases}$

remainder (Lagrange form): $|R_N(x)| = \left|\frac{x^{N+1}}{(N+1)!}\right| \underbrace{|f^{(N+1)}(t)|}_{\leq 1} \leq \frac{|x|^{N+1}}{(N+1)!} \xrightarrow{N\to\infty} 0 \quad \forall x \in \mathbb{R}$

(since even $\sum_{N=0}^{\infty} \frac{|x|^N}{N}$ bounded)

$$\Rightarrow \boxed{\sin x = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} (-1)^{\frac{k-1}{2}} \frac{x^k}{k!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \dots}$$

- similar: $\boxed{\cos x = \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} (-1)^{\frac{k}{2}} \frac{x^k}{k!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \dots}$

- $f(x) = e^{ix} \Rightarrow f(x) = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} = \sum_{k=0}^{\infty} i^k \frac{x^k}{k!}$

note: $i^k = \begin{cases} i & , k=1 \\ i^2=-1 & , k=2 \\ i^3=-i & , k=3 \\ i^4=1 & , k=4 \\ \vdots \end{cases} = \begin{cases} (-1)^{\frac{k}{2}} & , k \text{ even} \\ i(-1)^{\frac{k-1}{2}} & , k \text{ odd} \end{cases}$

$$\Rightarrow \boxed{e^{ix} = \underbrace{\sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} (-1)^{\frac{k}{2}} \frac{x^k}{k!}}_{= \cos x} + i \underbrace{\sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} (-1)^{\frac{k-1}{2}} \frac{x^k}{k!}}_{= \sin x} = \cos x + i \sin x}$$

we have recovered Euler's formula!

## Application: Newton's method

- we consider the iteration scheme $x_{N+1} = F(x_N)$
- we are looking for a fixed point, i.e., $z \in \mathbb{R}$ s.t. $z = F(z)$ $\quad (z = \lim_{N \to \infty} x_N)$
- recall Newton's method for finding zeros of $f$:

$$x_{N+1} = x_N - \frac{f(x_N)}{f'(x_N)}, \text{ i.e., } F(x) = x - \frac{f(x)}{f'(x)} \quad (\text{if } f(z)=0 \Rightarrow F(z)=z)$$

now we consider the error $\varepsilon_N = x_N - z$

$$\Rightarrow x_{N+1} = z + \varepsilon_{N+1} = F(x_N) = F(z + \varepsilon_N)$$

next time: Taylor expansion around $z$