Goal: approximate $f(x)$ near $b$ by a power series.

We compute: $\int_b^x f'(y) \, dy = f(x) - f(b)$

$$\Rightarrow f(x) = f(b) + \int_b^x f'(y) \, dy$$

Integration by parts: $\int_b^x f'(y) \cdot t \, dy = (y-x) f'(y) \bigg|_b^x - \int_b^x (y-x) f''(y) \, dy$

(recall: $h \cdot g = \int (hg)' = \int h'g + h \cdot g'$)

$$\Rightarrow \int_b^x (x-y) f''(y) \, dy = (x-b) f'(b) + \int_b^x (x-y) f''(y) \, dy$$

$$\Rightarrow f(x) = f(b) + (x-b) f'(b) + \int_b^x (x-y) f''(y) \, dy$$

Linear approximation

Another integration by parts yields (check the computation!):

$$f(x) = f(b) + (x-b) f'(b) + \frac{(x-b)^2}{2} f''(b) + \int_b^x \frac{(x-y)^2}{2} f'''(y) \, dy$$

Second order approximation (parabola)
Theorem (Taylor expansion):

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be $(N+1)$ times continuously differentiable on $[b, x]$. Then

$$f(x) = \sum_{k=0}^{N} \frac{(x-b)^k}{k!} f^{(k)}(b) + \int_{b}^{x} \frac{(x-y)^N}{N!} f^{(N+1)}(y) \, dy$$

$$=: R_N(x), \text{ called remainder}$$

Remarks:

- 3 other formulas for the remainder, e.g., the Lagrange form:

$$R_N(x) = \frac{f^{(N+1)}(t)}{(N+1)!} (x-b)^{N+1}$$

for some $t \in [b, x]$ (follows from mean-value theorem)

- $\sum_{k=0}^{\infty} \frac{(x-b)^k}{k!} f^{(k)}(b)$ is called Taylor series of $f$ around $b$

  If $b=0$, it is called Maclaurin series

- If $f$ is arbitrarily often differentiable and $R_N(x) \xrightarrow{N \to \infty} 0$ for some $x$ (near $b$), then

$$f(x) = \sum_{k=0}^{\infty} \frac{(x-b)^k}{k!} f^{(k)}(b)$$

- Note: The Taylor series for $f$ often differentiable $f$ might

  - converge to $f$ for some or all $x$
  - converge, but not to $f$ (if $R_N(x)$ does not converge to 0)
  - diverge (except at $x = b$)
Examples:

- \( f(x) = e^x \Rightarrow f^{(n)}(x) = e^x \Rightarrow f^{(k)}(0) = e^0 = 1 \)

  \[ \Rightarrow \text{Taylor series of } f \text{ around } b = 0 \text{ is } \sum_{k=0}^{\infty} \frac{x^k}{k!} \]

  We already know that indeed \( f(x) = \frac{\sum_{k=0}^{\infty} x^k}{k!} \) for all \( x \in \mathbb{R} \)

- \( f(x) = \sin x \Rightarrow f'(x) = \cos x \)
  \[ f''(x) = -\sin x \]
  \[ f'''(x) = -\cos x \]
  \[ f^{(4)}(x) = \sin x \]
  \[ \vdots \]

  (Let us choose \( b = 0 \): since \( \sin(0) = 0 \), \( \cos(0) = 1 \), we have \( f^{(k)}(0) = \begin{cases} 0 & \text{ if } k \text{ even } \\ (-1)^{\frac{k-1}{2}} & \text{ if } k \text{ odd } \end{cases} \)

  \[
  \text{remainder (Lagrange form): } |R_N(x)| = \left| \frac{x^{N+1}}{(N+1)!} \right| \frac{\left| f^{(N+1)}(t) \right|}{(N+1)!} \leq \frac{|x|^{N+1}}{(N+1)!} \xrightarrow{N \to \infty} 0 \quad \forall x \in \mathbb{R}
  \]

  \[
  \leq 1 \\
  \left( \text{since even } \sum_{k=0}^{\infty} \frac{|x|^k}{k!} \text{ is bounded} \right)
  \]

  \[ \Rightarrow \sin x = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{x^k}{k!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \ldots \]

  \[ \text{Similar: } \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{x^{2k}}{k!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \ldots \]
\[
\cdot f(x) = e^{ix} = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} = \sum_{k=0}^{\infty} \frac{i^k x^k}{k!}
\]

\[
\text{note: } i^k = \begin{cases} 
& i = 1, k = 1 \\
& i^2 = -1, k = 2 \\
& i^3 = -i, k = 3 \\
& i^4 = 1, k = 4 \\
& \vdots \\
& (-1)^{\frac{k}{2}} \quad i \text{ even} \\
& i(-1)^{\frac{k-1}{2}} \quad i \text{ odd}
\end{cases}
\]

\[
\rightarrow e^{ix} = \sum_{k=0}^{\infty} (-1)^{\frac{k}{2}} \frac{x^k}{k!} + i \sum_{k=1}^{\infty} (-1)^{\frac{k-1}{2}} \frac{x^k}{k!} = \cos x + i \sin x
\]

\[
\text{we have recovered Euler's formula!}
\]

**Application: Newton's method**

- we consider the iteration scheme \( x_{n+1} = F(x_n) \)
- we are looking for a fixed point, i.e., \( \exists z \in \mathbb{R} \text{ s.t. } z = F(z) \) \( (z = \lim_{n \to \infty} x_n) \)
- recall Newton's method for finding zeros of \( f \):
  \[
  x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ i.e., } F(x) = x - \frac{f(x)}{f'(x)} \quad (\text{if } f(z) = 0 \Rightarrow F(z) = z)
  \]

now we consider the error \( e_n = x_n - z \)

\[
\rightarrow x_{n+1} = z + e_{n+1} = F(x_n) = F(z + e_n)
\]

next time: Taylor expansion around \( z \)