Session 4
Feb. 12, 2020

last time:

- look for zeros of f, i.e., $f(z)=0$
- Newton's method: iteration $x_{N+1} = F(x_N)$, with $F(x) = x - \frac{f(x)}{f'(x)}$

$\Rightarrow$ we look for $\lim_{N\to\infty} x_N = z$ such that $f(z)=0$, i.e., $\boxed{z = F(z) \quad (\text{called "fixed point"})}$

now we consider the error $\varepsilon_N = x_N - z$

$$\Rightarrow x_{N+1} = z + \varepsilon_{N+1} = F(x_N) = F(z+\varepsilon_N)$$

now: second-order Taylor expansion: $F(x) = F(b) + (x-b)F'(b) + \frac{(x-b)^2}{2}F''(b) + R_2(x)$

we use $z=b$ and call $x = z+\varepsilon_N$ (s.t. $x-b=\varepsilon_N$)

$$\Rightarrow F(z+\varepsilon_N) = \underbrace{F(z)}_{=z} + \varepsilon_N F'(z) + \frac{\varepsilon_N^2}{2}F''(z) + \underbrace{\underbrace{R_2(z+\varepsilon_N)}_{= \dots \varepsilon_N^3 + \dots \varepsilon_N^4 + \dots}}_{\text{we assume this is small for small } \varepsilon_N}$$

above we also had $z + \varepsilon_{N+1} = F(z+\varepsilon_N)$

$$\Rightarrow z + \varepsilon_{N+1} = z + \varepsilon_N F'(z) + \frac{\varepsilon_N^2}{2}F''(z) + R_2$$

$$\Rightarrow \varepsilon_{N+1} = \varepsilon_N F'(z) + \frac{\varepsilon_N^2}{2}F''(z) + R_2$$

note: generally, if $F'(z) \neq 0$, we say we have linear convergence, meaning

$\varepsilon_{N+1} \approx \text{constant} \cdot \varepsilon_N$ (we neglect ε_N^2 and higher order terms, because they are very small for small ε_N)

but for Newton's method, we compute:

$$F'(x) = \left(x - \frac{f(x)}{f'(x)}\right)' = 1 - \underbrace{\left(\frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2}\right)}_{\text{quotient rule!}} = \frac{f(x)f''(x)}{(f'(x))^2}$$

$$\Rightarrow F'(z) = \frac{\overbrace{f(z)}^{=0} f''(z)}{(f'(z))^2} = 0 \quad \text{if } f'(z) \neq 0 \text{ which we need to assume}$$

$$\Rightarrow \varepsilon_{N+1} = \underbrace{\varepsilon_N^2 \frac{F''(z)}{2}}_{\text{leading order term}} + \underbrace{\dots \varepsilon_N^3 + \dots}_{\text{very small, neglected}}$$

$$\Rightarrow \boxed{\varepsilon_{N+1} \approx \text{constant} \cdot \varepsilon_N^2 \text{, which is called quadratic convergence}}$$

$\Rightarrow$ $\boxed{\text{Newton's method has quadratic speed of convergence}}$ (if it converges at all and $f'(z) \neq 0$)

note: speed of convergence very important for computer programs

<u>practical example:</u> suppose the constant above $= \frac{1}{2}$ and $\varepsilon_N = \frac{1}{10}$

↳ linear convergence: $\varepsilon_{N+1} = \frac{1}{2}\varepsilon_N = \frac{1}{2}\cdot\frac{1}{10}$, $\varepsilon_{N+2} = \frac{1}{2}\cdot\left(\frac{1}{2}\cdot\frac{1}{10}\right) = \frac{1}{40}, \dots$

↳ quadratic convergence: $\varepsilon_{N+1} = \frac{1}{2}\varepsilon_N^2 = \frac{1}{2}\cdot\left(\frac{1}{10}\right)^2$, $\varepsilon_{N+2} = \frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{10}\right)^2\right]^2 = \frac{1}{80000}, \dots$

$\Rightarrow$ MUCH faster

1.5 Improper Integrals

Definition: $\int_a^\infty f(x)\,dx := \lim_{b\to\infty} \int_a^b f(x)\,dx$, if it exists, is called improper integral

Ex.: $\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx = \lim_{b\to\infty} \left[-e^{-x}\right]_0^b = \lim_{b\to\infty} \left[-e^{-b} - (-e^{0})\right]$

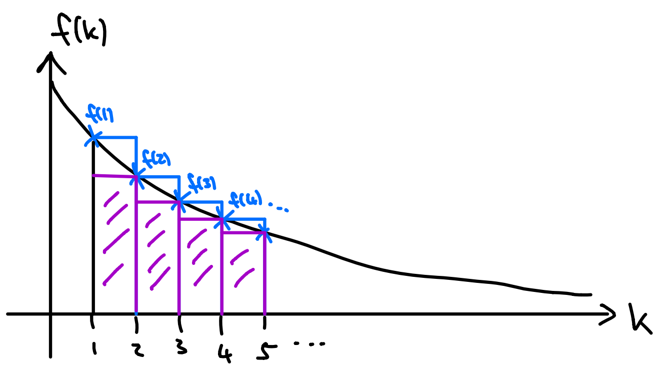
$$= \lim_{b\to\infty} \left[e^{-b} + 1\right] = 1$$

in short: $\int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1$

very useful application:

consider the series $\sum_{k=1}^{\infty} f(k) = f(1) + f(2) + f(3) + \ldots$ for some decreasing f.

Does it converge?



blue area $= \sum_{k=1}^{N} f(k) \geq \int_1^N f(x)\,dx$

purple area $= \sum_{k=2}^{N} f(k) \leq \int_1^N f(x)\,dx$

$\Rightarrow \sum_{k=2}^{N} f(k) \leq \int_1^N f(x)\,dx \leq \sum_{k=1}^{N} f(k)$

but $\lim\limits_{N\to\infty} \sum\limits_{k=2}^{N} f(k)$ exists if and only if $\lim\limits_{N\to\infty} \sum\limits_{k=1}^{N} f(k)$ exists

Conclusion:

Theorem (integral test):

Let $f(x) \geq 0 \ \forall x \geq 1$ and nonincreasing. Then

$$\sum_{k=1}^{\infty} f(k) \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges}$$

Example: • does $\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ converge? $\Rightarrow f(x) = \frac{1}{x^2}$

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{\infty} = 0 - \left(-\frac{1}{1}\right) = 1$$, so answer is yes!

• does $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ converge? $\Rightarrow f(x) = \frac{1}{x}$

$$\int_1^{\infty} \frac{1}{x}\,dx = [\ln x]_1^{\infty} = \underbrace{\ln\infty}_{\text{diverges!}} - \underbrace{\ln 1}_{=0}$$, so answer is no!