More on surfaces:

nice visualizations: geogebra.org

3D Calculator

Example from last time: \( f(x,y) = x \sin y \)

- tangent plane at \( a = (a_x, a_y) \) is 
  \[
  z = a_x \sin(a_x) + \sin(a_z)(x-a_x) + a_z \cos(a_z)(y-a_z)
  \]

- e.g., at \( a = \left( \frac{\pi}{2} \right) \), we have 
  \[
  z = 1 \cdot \sin \left( \frac{\pi}{2} \right) + \sin \left( \frac{\pi}{2} \right)(x-1) + 1 \cdot \cos \left( \frac{\pi}{2} \right)(y-\frac{\pi}{2})
  \]
  \[
  = 1 + (x-1)
  \]

\[
\begin{pmatrix}
  \sin x \\
  \cos x
\end{pmatrix}
\]

\[
\begin{pmatrix}
  -1 & 0 & 1 \\
  0 & 1 & -1 \\
  1 & -1 & 0
\end{pmatrix}
\]

Note (recall): There are at least two ways to represent surfaces (hence, 2-dimensional):

- graph of a function: \( \{ (x,y,z) \in \mathbb{R}^3 : x \in \mathbb{R}, y \in \mathbb{R} \} \)

  \( \Rightarrow \) only one \( z \)-value for each \( (x,y) \)

  Ex.: upper half-sphere: \( f(x,y) = \sqrt{4-x^2-y^2} \) for \( 0 \leq x^2 + y^2 \leq 4 \)

- solution to equation \( F(x,y,z) = 0 \): \( \{ (x,y,z) \in \mathbb{R}^3 : F(x,y,z) = 0 \} \)

  \( \rightarrow \) several \( z \)-values possible given some \( (x,y) \)

  Ex.: sphere: \( F(x,y,z) = x^2 + y^2 + z^2 - 4 = 0 \)
For the special case of tangent planes, we have:

- tangent plane at \( \vec{a} = (a_1, a_2) \) = graph of \( z(x,y) = f(\vec{a}) + \left( \frac{\partial_x f(\vec{a})}{\partial_y f(\vec{a})} \right) (x - a_1) \)

\[ = (\nabla f)(\vec{a}) \cdot (\vec{x} - \vec{a}) \]  
(2 dim. scalar product)

- let us write \( \vec{X} = (x, y), \vec{A} = (a_1, a_2) \) with \( a_3 = f(a_1, a_2) \) and \( F(x,y,z) = f(x,y) - z \), such that surface (= graph of \( f \)) is solution to \( F(x,y,z) = 0 \)

\[ \Rightarrow \text{equation for tangent plane becomes } (\nabla F)(\vec{A}) \cdot (\vec{X} - \vec{A}) = 0 \]

3-dim. scalar product

check: \( (\nabla F)(\vec{A}) = \begin{pmatrix} \frac{\partial_x f}{\partial x} & \frac{\partial_y f}{\partial x} \\ \frac{\partial_x f}{\partial y} & \frac{\partial_y f}{\partial y} \end{pmatrix} (\vec{A}) \), and \( (\vec{X} - \vec{A}) = \begin{pmatrix} x - a_1 \\ y - a_2 \\ z \end{pmatrix} \) ✔

back to differentiability ...

recall what we defined so far for \( f: \mathbb{R}^n \rightarrow \mathbb{R} \)

- \( \frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(\vec{x} + h\vec{e}_i) - f(\vec{x})}{h} \), partial derivative

- for a unit vector \( \vec{u} \) (i.e., \( \|\vec{u}\| = 1 \)), \( D_{\vec{u}} f = \lim_{h \to 0} \frac{f(\vec{x} + h\vec{u}) - f(\vec{x})}{h} \), directional derivative

- \( f \) (totally) differentiable at \( \vec{a} \) if \( f(\vec{a} + \vec{h}) = f(\vec{a}) + \vec{u} \cdot \vec{h} + E(\vec{h}) \) with \( \lim_{h \to 0} \frac{E(\vec{h})}{\|\vec{h}\|} = 0 \)

for some \( \vec{u} \).

Note: \( \vec{u} \) is called the (total) derivative of \( f \) at \( \vec{a} \).

let us give here (without proof) the connections between these notions:
Theorem:
- If $f$ is differentiable at $\mathbf{a}$, then all partial derivatives exist, and the total derivative is the gradient (i.e., $\mathbf{m} = (\nabla f)(\mathbf{a})$).
- $f$ continuously differentiable at $\mathbf{a} \iff$ all partial derivatives at $\mathbf{a}$ exist and are continuous.

Note: 3 examples where all partial derivatives exist, but the function is not differentiable,

\[ f(x,y) = \begin{cases} \frac{x^2 y^2}{x^2 + y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases} \]

(bonus problem on HW)

"One cannot sensibly put a tangent plane at the origin $(0,0)$".

- If all partial derivatives exist and are continuous, $f$ is called a $C^1$-function (or "of class $C^1$").. If all combinations of $k$ partial derivatives exist and are continuous, $f$ is called $C^k$-function.

Theorem: If $f$ is differentiable at $\mathbf{a}$, then the directional derivatives $(D_{\mathbf{u}} f)(\mathbf{a})$ exist for all $\mathbf{u} \in \mathbb{R}^n$, $|\mathbf{u}| = 1$, and

\[ (D_{\mathbf{u}} f)(\mathbf{a}) = (\nabla f)(\mathbf{a}) \cdot \mathbf{u}. \]

Proof: $f$ differentiable at $\mathbf{a}$ means

\[ \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a}) - (\nabla f)(\mathbf{a}) \cdot h}{|h\mathbf{u}|} = 0 \]

now (let $h = t \mathbf{u}$, $t > 0$)

\[ \lim_{t \to 0} \frac{f(\mathbf{a} + t\mathbf{u}) - f(\mathbf{a}) - (\nabla f)(\mathbf{a}) \cdot t}{t} = \lim_{t \to 0} \frac{f(\mathbf{a} + t\mathbf{u}) - f(\mathbf{a})}{t} - (\nabla f)(\mathbf{a}) \cdot \mathbf{u} \]

Note: $f(x,y) = \begin{cases} \frac{x^2 y^2}{x^2 + y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$ is an example where all directional derivatives at $(0,0)$ exist, but $f$ is not differentiable at $(0,0)$. 
Geometric interpretation for $(\nabla f)(\vec{a}) \neq 0$:

- Recall that $(\nabla f)(\vec{a}) \cdot \vec{u} = |(\nabla f)(\vec{a})| \frac{1}{\sqrt{\cos \theta}} \quad (\text{where } \theta = \text{angle between } \nabla f(\vec{a}) \text{ and } \vec{u})$

- Directional derivative maximal if $\vec{u}$ points in same direction as $(\nabla f)(\vec{a})$

$\Rightarrow (\nabla f)(\vec{a})$ points in direction of largest directional derivative

Example from last time: $f(x,y) = x^2 + 5x$, $\vec{u} = \frac{1}{\sqrt{2}} (1,1)$, $\vec{u} = 1$ (s.t. $|\vec{u}| = 1$)

$\Rightarrow (\nabla f)(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \right) = \frac{1}{\sqrt{2}} \left( 2x + 5 + x^2 \right)$, same as last time

Next: Chain rule

Consider $x_1(t), \ldots, x_n(t)$, and a function $f(x_1,t), \ldots, x_n(t))$

Heuristic computation: the differential is $df = \frac{\partial f}{\partial x_1} dx_1 + \ldots + \frac{\partial f}{\partial x_n} dx_n$

$\Rightarrow$ $\frac{df}{dt} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \ldots + \frac{\partial f}{\partial x_n} \frac{dx_n}{dt}$

**Theorem:** Let $f$ be differentiable at $\vec{x} = \vec{b}$ and $x_1(t), \ldots, x_n(t)$ be differentiable at $t = a$, $\vec{b} = \vec{x}(a)$, $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$. Then $\frac{df}{dt}(a) = (\nabla f)(\vec{b}) \cdot \left( \frac{d\vec{x}}{dt} \right)(a)$

$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{dx_i}{dt}(a)$.
Note: \( \dot{x}(t) \) could, e.g., be the trajectory of a particle

\[ \Rightarrow \frac{d\dot{x}}{dt} \text{ is the velocity} \]

\[ \text{Ex.}: \ f(x_1, x_2) = x_1e^{-x_2}, \ x_1(t) = t^2, \ x_2(t) = t^3 \]

\[ \Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dt} = e^{-x_2(t)} 2t - x_1(t)e^{-x_2(t)} 3t^2 \]

\[ = (2t - 3t^4)e^{-t^3} \]

(Note: \( \frac{df}{dt} \) is here the usual one-dimensional derivative, so in this simple example we also could have computed explicitly)

\[ \frac{d}{dt} f(x_1(t), x_2(t)) = \frac{d}{dt} \left(t^2e^{-t^3}\right) = 2te^{-t^3} + t^2(-3t^2)e^{-t^3} = (2t - 3t^4)e^{-t^3} \]