Session 7
Feb. 24, 2020

More on surfaces:

nice visualizations: geogebra.org
↳ 3D Calculator

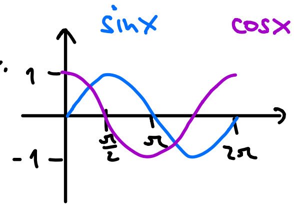
Example from last time: $f(x,y) = x\sin(y)$

↳ tangent plane at $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is $z = a_1 \sin(a_2) + \sin(a_2)(x - a_1) + a_1 \cos(a_2)(y - a_2)$

↳ e.g., at $\vec{a} = \begin{pmatrix} 1 \\ \frac{\pi}{2} \end{pmatrix}$, we have $z = 1 \cdot \sin(\frac{\pi}{2}) + \sin(\frac{\pi}{2})(x-1) + 1 \cdot \cos(\frac{\pi}{2})(y - \frac{\pi}{2})$

$= 1 + (x-1)$

$= x$

(recall: 

)

Note/recall: $\exists$ at least two ways to represent surfaces (here, 2 dimensional)

- graph of a function: $\left\{ \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix} \in \mathbb{R}^3 : x \in \mathbb{R}, y \in \mathbb{R} \right\}$

  ↳ only <u>one</u> z-value for each (x,y)

  Ex.: upper half-sphere: $f(x,y) = \sqrt{4 - x^2 - y^2}$ for $0 \le x^2 + y^2 \le 4$

- solution to equation $F(x,y,z) = 0$: $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : F(x,y,z) = 0 \right\}$

  ↳ several z possible given some (x,y)

  Ex.: sphere: $F(x,y,z) = x^2 + y^2 + z^2 - 4 = 0$

For the special case of tangent planes, we have:

- tangent plane at $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ = graph of $\boxed{z(x,y) = f(\vec{a}) + \underbrace{\begin{pmatrix} \partial_x f(\vec{a}) \\ \partial_y f(\vec{a}) \end{pmatrix} \cdot \begin{pmatrix} x - a_1 \\ y - a_2 \end{pmatrix}}}$

$$= (\vec{\nabla} f)(\vec{a}) \cdot (\vec{x} - \vec{a})$$

(2dim. scalar product)

- let us write $\vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\vec{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ with $a_3 = f(a_1, a_2)$, and $F(x,y,z) = f(x,y) - z$,

  such that surface (= graph of f) is solution to $F(x,y,z) = 0$

  $\Rightarrow$ equation for tangent plane becomes $\boxed{(\vec{\nabla} F)(\vec{A}) \underbrace{\cdot}_{} (\vec{X} - \vec{A}) = 0}$

  3-dim. scalar product

  check: $(\vec{\nabla} F)(\vec{A}) = \begin{pmatrix} \partial_x f \\ \partial_y f \\ \partial_z (-z) \end{pmatrix} (\vec{A}) = \begin{pmatrix} \partial_x f \\ \partial_y f \\ -1 \end{pmatrix} (\vec{A})$, and $(\vec{X} - \vec{A}) = \begin{pmatrix} x - a_1 \\ y - a_2 \\ z - f(a_1, a_2) \end{pmatrix}$ ✓

back to differentiability ...

recall what we defined so far for $f: \mathbb{R}^n \to \mathbb{R}$

- $\dfrac{\partial f}{\partial x_i} = \lim\limits_{h \to 0} \dfrac{f(\vec{x} + h\vec{e}_i) - f(\vec{x})}{h}$, partial derivative

- for a unit vector $\vec{u}$ (i.e., $|\vec{u}| = 1$), $D_{\vec{u}} f := \lim\limits_{h \to 0} \dfrac{f(\vec{x} + h\vec{u}) - f(\vec{x})}{h}$, directional derivative

- f (totally) differentiable at $\vec{a}$ if $f(\vec{a} + \vec{h}) = f(\vec{a}) + \vec{m} \cdot \vec{h} + E(\vec{h})$ with $\lim\limits_{h \to 0} \dfrac{E(\vec{h})}{|\vec{h}|} = 0$

  for some $\vec{m}$.

  Note: $\boxed{\vec{m} \text{ is called the (total) derivative of } f \text{ at } \vec{a}.}$

Let us give here (without proof) the connections between these notions:

Theorem:

- If f differentiable at $\vec{a}$, then all partial derivatives exist, and the total derivative is the gradient (i.e., $\vec{m} = (\vec{\nabla} f)(\vec{a})$).
- f <u>continuously</u> differentiable at $\vec{a}$ $\Longleftrightarrow$ all partial derivatives at $\vec{a}$ exist and are <u>continuous</u>

Note: • $\exists$ examples where all partial derivatives exist, but the function is not differentiable,

e.g., $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$ (bonus problem on HW)

"one cannot sensibly put a tangent plane at the origin $(0,0)$"

• If all partial derivatives exist and are continuous, f is called a C^1-function (or "of class C^1"). If all combinations of k partial derivatives exist and are continuous, f is called C^k-function.

Theorem: If f differentiable at $\vec{a}$, then the directional derivatives $(D_{\vec{u}} f)(\vec{a})$ exist for all $\vec{u} \in \mathbb{R}^n$, $|\vec{u}| = 1$, and $\boxed{(D_{\vec{u}} f)(\vec{a}) = (\vec{\nabla} f)(\vec{a}) \cdot \vec{u}}$.

Proof: f differentiable at $\vec{a}$ means $\underbrace{\frac{f(\vec{a}+\vec{h}) - f(\vec{a}) - (\vec{\nabla} f)(\vec{a}) \cdot \vec{h}}{|\vec{h}|}} \xrightarrow{h \to 0} 0$

now let $\vec{h} = t\vec{u}$ $(t>0)$ $\Longrightarrow$ $= \frac{f(\vec{a}+t\vec{u}) - f(\vec{a}) - (\vec{\nabla} f)(\vec{a}) \cdot \vec{u}\, t}{t \underbrace{|\vec{u}|}_{=1}} = \underbrace{\frac{f(\vec{a}+t\vec{u}) - f(\vec{a})}{t}} - (\vec{\nabla} f)(\vec{a}) \cdot \vec{u}$

$\xrightarrow{t \to 0} (D_{\vec{u}} f)(\vec{a})$ $\square$

Note: $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$ is an example where all directional derivatives at $(0,0)$ exist, but f is not differentiable at $(0,0)$.

Geometric interpretation for $(\vec{\nabla} f)(\vec{a}) \neq 0$:

$\hookrightarrow$ recall that $(\vec{\nabla} f)(\vec{a}) \cdot \vec{u} = |(\vec{\nabla} f)(\vec{a})| \underbrace{|\vec{u}|}_{=1} \cos\theta$ (θ = angle between $(\vec{\nabla} f)(\vec{a})$ and $\vec{u}$)

$\Rightarrow$ directional derivative maximal if if $\vec{u}$ points in same direction as $(\vec{\nabla} f)(\vec{a})$

$\Rightarrow$ $\boxed{(\vec{\nabla} f)(\vec{a}) \text{ points in direction of largest directional derivative}}$

Example from last time: $f(x,y) = x^2 y + 5x$, $\vec{u} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ (s.t. $|\vec{u}| = 1$)

$$\Rightarrow (D_{\vec{u}} f)(x,y) = \begin{pmatrix}\partial_x f\\ \partial_y f\end{pmatrix} \cdot \begin{pmatrix}1\\1\end{pmatrix} \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\left(\partial_x f + \partial_y f\right) = \frac{1}{\sqrt{2}}\left(2xy + 5 + x^2\right),$$ same as last time

Next: chain rule

consider $x_1(t), \ldots, x_n(t)$, and a function $f(x_1(t), \ldots, x_n(t))$

heuristic computation: the differential is $df = \frac{\partial f}{\partial x_1} dx_1 + \ldots + \frac{\partial f}{\partial x_n} dx_n$

$$\Rightarrow \boxed{\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \ldots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}}$$

<u>Theorem:</u> Let f be differentiable at $\vec{x} = \vec{b}$ and $x_1(t), \ldots, x_n(t)$ be differentiable at $t = a$, $\vec{b} = \vec{x}(a)$, $\vec{x}(t) = \begin{pmatrix}x_1(t)\\ \vdots \\ x_n(t)\end{pmatrix}$. Then

$$\frac{df}{dt}(a) = (\vec{\nabla} f)(\vec{b}) \cdot \left(\frac{d\vec{x}}{dt}\right)(\vec{a})$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{b}) \frac{dx_i}{dt}(a).$$

Note: • $\vec{x}(t)$ could, e.g., be the trajectory of a particle

$\Rightarrow \frac{d\vec{x}}{dt}$ is the velocity

Ex.: $f(x_1,x_2) = x_1 e^{-x_2}$, $x_1(t) = t^2$, $x_2(t) = t^3$

$$\Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} = e^{-x_2(t)}\,2t - x_1(t)e^{-x_2(t)}\,3t^2$$

$$= \left(2t - 3t^4\right)e^{-t^3}$$

$\Bigg($ Note: $\frac{df}{dt}$ is here the usual one-dimensional derivative, so in this simple example we

also could have computed explicitly

$$\frac{d}{dt} f(x_1(t), x_2(t)) = \frac{d}{dt}\left(t^2 e^{-t^3}\right) = 2te^{-t^3} + t^2(-3t^2)e^{-t^3} = (2t-3t^4)e^{-t^3} \Bigg)$$