Recall: for \( f(x_1(t), \ldots, x_n(t)) \) the chain rule reads
\[
\frac{df}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{dx_i}{dt} = \left( \nabla f \right) \cdot \frac{dx}{dt}
\]

**Potential energy**

\[
\frac{\Delta V}{\Delta t} = \nabla V \cdot \Delta \mathbf{r}
\]

**Kinetic energy**

Ex: classical mechanics: \( H(q, \dot{q}, p) = \frac{p^2}{2} + V(q) \) (mass \( m = 1 \))

\( \dot{q} = \text{particle position}, \dot{p} = \frac{dq}{dt} = \text{momentum/velocity} \)

Newton's law reads \( \frac{d\dot{q}}{dt} = -\nabla V(q) \) (force = mass \( \cdot \) acceleration)

\[
\frac{dH}{dt} = \left( \nabla_q H \right) \cdot \dot{q} + \left( \nabla_p H \right) \cdot \dot{p}
\]

\[
= \left( \nabla_q H \right) \cdot \dot{q} + \left( \nabla_p H \right) \cdot \dot{p} - \frac{dV}{dq}
\]

\[
= \left( \nabla_q H \right) \cdot \dot{q} + \left( \nabla_p H \right) \cdot \dot{p} + \left( \nabla_p H \right) \cdot \frac{dV}{dq}
\]

\( = 0 \), which is energy conservation!

Remark: if \( X_i(t_1, \ldots, t_m) \), then we have \( \frac{\partial f}{\partial t_k} = \sum_{i=1}^{k} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{dt_k} \)

Next: a few more topics/applications of many-variable derivatives
A) More on differentials

force $\vec{F}$, potential $V$ are related by $\vec{F} = -\nabla V$

- given $V$, we can directly compute $\vec{F}$
- but given $\vec{F}$, can we find a $V$?

In other words: given a differential $dV = \frac{\partial V}{\partial x_1} dx_1 + \cdots + \frac{\partial V}{\partial x_n} dx_n$, can we find $V$?

If yes, we call $dV$ an **exact differential**; if no, $dV$ is called **inexact**.

**Ex.**

\[
\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} dy = f(x,y) = xy + c \quad (c \in \mathbb{R}, \text{some constant})
\]

\[
\implies \text{exact differential}
\]

\[
\frac{df}{dy} = 3y dx + x dy
\]

\[
\implies \text{need } f(x,y) = 3x y + g(y), \text{ for some fct. } g
\]

\[
\implies \frac{df}{dy} = 3x + \frac{dg(y)}{dy} \implies \text{inexact differential}
\]
General answer: consider \( df = A(x, y)dx + B(x, y)dy \)
\[
= \frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}
\]

\[
\Rightarrow \quad \frac{\partial}{\partial y} \frac{\partial f}{\partial x} = \frac{\partial A}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial B}{\partial x}
\]

but these are equal! (due to Clairaut’s Schwarz (see Session 5))

\[
\Rightarrow \quad \text{need} \quad \frac{\partial A}{\partial y} = \frac{\partial B}{\partial x} \quad \text{as a necessary condition}
\]

It turns out (here without proof) that this is also sufficient!

Generally: \( df = \sum_{i=1}^{n} g_i(x_1, \ldots, x_n) \, dx_i \) is exact

\[
\iff \quad \frac{\partial g_i}{\partial x_k} = \frac{\partial g_k}{\partial x_i} \quad \forall i, k = 1, \ldots, n
\]

B) Taylor series

Consider the function \( g(t) = f(a + t \, h) \)

What is the Taylor expansion of \( g \) around 0 at \( t = 1 \)?

(\( \text{this will lead us to an expansion of } f(a + h) \))
\[ g'(t) = (\vec{V} f)(\vec{a} + \vec{u}) \cdot \frac{d(\vec{a} + \vec{u})}{dt} = \vec{v} \cdot (\vec{V} f)(\vec{a} + \vec{u}) = h_1 \partial_{x_1} f + \ldots + h_n \partial_{x_n} f = (h_1 \partial_{x_1} + \ldots h_n \partial_{x_n}) f = (\hat{\vec{v}} \cdot \vec{V}) f(\vec{a} + \vec{u}) \]

\[ g''(t) = (\hat{\vec{v}} \cdot \vec{V}) (\hat{\vec{v}} \cdot \vec{V}) f(\vec{a} + \vec{u}) \]

\[ g^{(k)}(t) = (\hat{\vec{v}} \cdot \vec{V})^k f(\vec{a} + \vec{u}) \]

\[ \Rightarrow \text{Taylor expansion} \quad g(t) = \sum_{k=0}^{N} \frac{t^k}{k!} g^{(k)}(0) + R_N(t) = (\hat{\vec{v}} \cdot \vec{V})^k f(\vec{a}) \]

\[ g(1) = \sum_{k=0}^{N} \frac{(\hat{\vec{v}} \cdot \vec{V})^k f(\vec{a})}{k!} + R_N(1) \]

\[ = f(\vec{a} + \vec{u}) \]

\[ \Rightarrow f(\vec{a} + \vec{u}) = \sum_{k=0}^{\infty} \frac{(\hat{\vec{v}} \cdot \vec{V})^k f(\vec{a})}{k!} \] is the infinite Taylor series of \( f \) around \( \vec{a} \)

\[ \text{Note: there are expressions for the rest term, but they are a bit lengthy, so let's not write them down here.} \]
Ex.: Second-order expansion of \( f(x_1, x_2) \) around \( \bar{a} = (\bar{a}_x, \bar{a}_y) \); call \( \Delta \tilde{a} = (\Delta x, \Delta y) \)

\[
\Rightarrow f(\bar{a} + (\Delta x, \Delta y)) = f(\bar{a}) + (\Delta \tilde{a} \cdot \nabla) f(\bar{a}) + \frac{1}{2} (\Delta \tilde{a} \cdot \nabla)(\Delta \tilde{a} \cdot \nabla) f(\bar{a}) + \text{Rest}
\]

\[
= (\Delta x \frac{\partial f}{\partial x} + \Delta y \frac{\partial f}{\partial y}) f(\bar{a})
\]

\[
= \Delta x \frac{\partial f}{\partial x}(\bar{a}) + \Delta y \frac{\partial f}{\partial y}(\bar{a})
\]

\[
\Rightarrow f(\bar{a} + (\Delta x, \Delta y)) = f(\bar{a}) + \Delta x \frac{\partial f}{\partial x}(\bar{a}) + \Delta y \frac{\partial f}{\partial y}(\bar{a})
\]

\[
+ \frac{1}{2} \Delta x^2 \frac{\partial^2 f}{\partial x^2}(\bar{a}) + \frac{1}{2} \Delta y^2 \frac{\partial^2 f}{\partial y^2}(\bar{a}) + \Delta x \Delta y \frac{\partial^2 f}{\partial x \partial y}(\bar{a}) + \text{Rest}
\]

C) Change of variables

Sometimes we would like to express functions \( f(x_1, x_2) \) in terms of different variables.

Ex.: polar coordinates

\[
x_1 = r \cos \varphi, \quad x_2 = r \sin \varphi \quad (r \geq 0, \ \varphi \in [0, 2\pi])
\]
\[
\text{chain rule tells us that } \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial r} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial r}
\]

e.g., for \( f(x_1, x_2) = \sqrt{x_1^2 + x_2^2} \)

\[
\Rightarrow f(r, \theta) = \sqrt{(r \cos \theta)^2 + (r \sin \theta)^2} = r, \quad \text{linear fct. in } r
\]

\[
\Rightarrow \frac{\partial f}{\partial r} = 1 \quad \text{(much faster than computing a directional derivative)}
\]