Session 8
Feb. 26, 2020

recall: for $f(x_1(t), \dots, x_n(t))$ the chain rule reads

$$\boxed{\frac{df}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{dx_i}{dt} = (\vec{\nabla} f) \cdot \frac{d\vec{x}}{dt}}$$

potential energy

Ex.: classical mechanics: $H(\vec{q}, \vec{p}) = \frac{\vec{p}^2}{2} + V(\vec{q})$ (mass $m = 1$)

kinetic energy

$\vec{q}$ = particle position, $\vec{p} = \frac{d\vec{q}}{dt}$ momentum/velocity

Newton's law reads $\frac{d\vec{p}}{dt} = -\vec{\nabla} V(\vec{q})$ (force = mass · acceleration)

chain rule

$$\frac{dH}{dt} = (\vec{\nabla}_{\vec{q}} H) \cdot \frac{d\vec{q}}{dt} + (\vec{\nabla}_{\vec{p}} H) \cdot \frac{d\vec{p}}{dt}$$

$$= (\vec{\nabla}_{\vec{q}} H) \cdot \vec{p} + (\vec{\nabla}_{\vec{p}} H)(-\vec{\nabla}_{\vec{q}} V(\vec{q}))$$

$$= (\vec{\nabla}_{\vec{q}} H) \cdot (\vec{\nabla}_{\vec{p}} H) + (\vec{\nabla}_{\vec{p}} H)(-\nabla_{\vec{q}} H)$$

$= 0$, which is energy conservation!

Remark: If $x_i(t_1, \dots, t_m)$, then we have $\frac{\partial f}{\partial t_k} = \sum_{i=1}^{k} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial t_k}$

next: a few more topics/applications of many-variable derivatives

A) More on differentials

force $\vec{F}$, potential V are related by $\vec{F} = -\vec{\nabla} V$

- given V, we can directly compute $\vec{F}$
- but given $\vec{F}$, can we find a V?

In other words: given a differential $dV = \frac{\partial V}{\partial x_1} dx_1 + \ldots + \frac{\partial V}{\partial x_n} dx_n$, can we find V?

If yes, we call dV an exact differential, if no dV is called inexact.

Ex.:
- $df = \underbrace{y}_{\frac{\partial f}{\partial x}} dx + \underbrace{x}_{\frac{\partial f}{\partial y}} dy \implies f(x,y) = xy + c$ ($c \in \mathbb{R}$ some constant)

  $\implies$ exact differential

- $df = \underbrace{3y}_{\frac{\partial f}{\partial x}} dx + \underbrace{x}_{\frac{\partial f}{\partial y}} dy$

  $\hookrightarrow$ need $f(x,y) = 3xy + g(y)$ for some fct. g

  $\implies \frac{\partial f}{\partial y} = \underbrace{3x}_{\text{factor 3 too much}} + \underbrace{\frac{dg(y)}{dy}}_{\text{fct. of } y} \implies$ inexact differential

General answer: consider $df = \underbrace{A(x,y)}_{=\frac{\partial f}{\partial x}}dx + \underbrace{B(x,y)}_{\frac{\partial f}{\partial y}}dy$

$$\Rightarrow \underbrace{\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial A}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial x}\frac{\partial f}{\partial y}}_{} = \frac{\partial B}{\partial x}$$

but these are equal! (due to Clairaut/Schwarz (see Session 5))

$\Rightarrow$ need $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$ as a necessary condition

It turns out (here without proof) that this is also sufficient!

$$\boxed{\begin{array}{l}\text{Generally: } df = \sum_{i=1}^{n} g_i(x_1,\dots,x_n)\,dx_i \text{ is exact} \\ \qquad \Leftrightarrow \frac{\partial g_i}{\partial x_k} = \frac{\partial g_k}{\partial x_i} \quad \forall\, i,k = 1,\dots,n\end{array}}$$

B) Taylor series

consider the function $\underbrace{g(t)}_{\text{one variable}} = f(\underbrace{\vec{a}+t\vec{h}}_{\text{many variables}})$

What is the Taylor expansion of g around 0 at $t=1$?

(this will lead us to an expansion of $f(\vec{a}+\vec{h})$)

$$\Rightarrow \underset{\substack{\uparrow \\ \text{chain rule}}}{g'(t) =} (\vec{\nabla} f)(\vec{a}+t\vec{h}) \cdot \underbrace{\frac{d(\vec{a}+t\vec{h})}{dt}}_{=\vec{h}} = \underbrace{\vec{h} \cdot (\vec{\nabla} f)(\vec{a}+t\vec{h})}$$

$$= h_1 \partial_{x_1} f + \ldots + h_n \partial_{x_n} f$$

$$= (h_1 \partial_{x_1} + \ldots h_n \partial_{x_n}) f$$

$$= (\vec{h} \cdot \vec{\nabla}) f(\vec{a}+t\vec{h})$$

$$\Rightarrow g''(t) = (\vec{h} \cdot \vec{\nabla})(\vec{h} \cdot \vec{\nabla}) f(\vec{a}+t\vec{h})$$

$$\Rightarrow g^{(k)}(t) = (\vec{h} \cdot \vec{\nabla})^k f(\vec{a}+t\vec{h})$$

$\Rightarrow$ Taylor expansion $g(t) = \sum_{k=0}^{N} \frac{t^k}{k!} \underbrace{g^{(k)}(0)}_{=(\vec{h}\cdot\vec{\nabla})^k f(\vec{a})} + R_N(t)$

$$\Rightarrow \underset{\substack{\parallel \\ f(\vec{a}+\vec{h})}}{g(1)} = \sum_{k=0}^{N} \frac{(\vec{h}\cdot\vec{\nabla})^k f(\vec{a})}{k!} + R_N(1)$$

$\Rightarrow \boxed{f(\vec{a}+\vec{h}) = \sum_{k=0}^{\infty} \frac{(\vec{h}\cdot\vec{\nabla})^k f(\vec{a})}{k!}}$ is the infinite Taylor series of f around $\vec{a}$

Note: there are expressions for the rest term, but they are a bit lengthy, so let's not write them down here.

Ex.: Second-order expansion of $f(x_1, x_2)$ around $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$; call $\vec{h} = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$

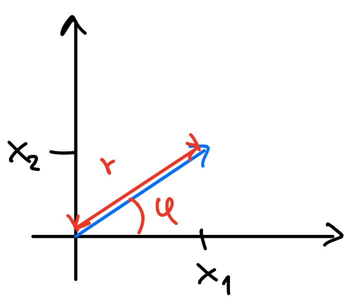
$$\Rightarrow f\left(\vec{a} + \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}\right) = f(\vec{a}) + \underbrace{(\vec{h}\cdot\vec{\nabla})}_{\substack{= \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} f \\ = \Delta x \frac{\partial f}{\partial x}(\vec{a}) + \Delta y \frac{\partial f}{\partial y}(\vec{a})}} f(\vec{a}) + \frac{1}{2} \underbrace{(\vec{h}\cdot\vec{\nabla})(\vec{h}\cdot\vec{\nabla})}_{= (\Delta x \partial_x + \Delta y \partial_y)(\Delta x \partial_x + \Delta y \partial_y)} f(\vec{a}) + \text{Rest}$$

$$\boxed{\begin{aligned} \Rightarrow f\left(\vec{a} + \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}\right) &= f(\vec{a}) + \Delta x \frac{\partial f(\vec{a})}{\partial x} + \Delta y \frac{\partial f(\vec{a})}{\partial y} \\ &\quad + \frac{1}{2} \Delta x^2 \frac{\partial^2 f(\vec{a})}{\partial x^2} + \frac{1}{2} \Delta y^2 \frac{\partial^2 f(\vec{a})}{\partial y^2} + \Delta x \Delta y \frac{\partial^2 f(\vec{a})}{\partial x \partial y} + \text{Rest} \end{aligned}}$$

c) Change of variables

sometimes we would like to express functions $f(x_1, x_2)$ in terms of different variables

Ex.: polar coordinates



instead of writing $f(x_1, x_2)$, we would like write $f(r, \varphi)$ with

$$x_1 = r\cos\varphi, \quad x_2 = r\sin\varphi \quad (r \geq 0,\ \varphi \in [0, 2\pi))$$

$\Rightarrow$ chain rule tells us that $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial r} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial r}$

e.g., for $f(x_1,x_2) = \sqrt{x_1^2+x_2^2}$

$$\Rightarrow f(r,\varphi) = \sqrt{(r\cos\varphi)^2+(r\sin\varphi)^2} = r\text{, linear fct. in } r$$

$\Rightarrow \frac{\partial f}{\partial r} = 1$ (much faster than computing a directional derivative)