2.6 Vector Operators

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Certain common operations involving partial derivatives have names.

As before, we use $\vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$ as a vector that can operate on functions in different ways:

- For $\varphi: \mathbb{R}^n \to \mathbb{R}$ (a scalar field), $\vec{\nabla}\varphi = \begin{pmatrix} \frac{\partial \varphi}{\partial x_1} \\ \vdots \\ \frac{\partial \varphi}{\partial x_n} \end{pmatrix} =: \operatorname{grad}\varphi$ is called $\boxed{\text{gradient of } \varphi}$.

- For $\vec{f}: \mathbb{R}^n \to \mathbb{R}^n$ (a vector field), $\underbrace{\vec{\nabla}\cdot\vec{f}}_{\text{scalar product}} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} =: \operatorname{div}\vec{f}$

  is called $\boxed{\text{divergence of } \vec{f}}$.

- For $\vec{f}: \mathbb{R}^3 \to \mathbb{R}^3$ (a vector field), $\underbrace{\vec{\nabla}\times\vec{f}}_{\text{cross product}} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} \times \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} \\ \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \end{pmatrix} =: \operatorname{curl}\vec{f}$

  is called $\boxed{\text{curl of } \vec{f}}$.

These operations get really interesting in the context of line/surface/volume integrals (Gauss and Stokes theorems), but we don't have the time to go into this.

Let us here just note some interesting identities:

- $$\operatorname{curl}\operatorname{grad}\varphi = \vec{\nabla}\times(\vec{\nabla}\varphi) = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix} \times \begin{pmatrix} \partial_{x_1}\varphi \\ \partial_{x_2}\varphi \\ \partial_{x_3}\varphi \end{pmatrix} = \begin{pmatrix} \partial_{x_2}\partial_{x_3}\varphi - \partial_{x_3}\partial_{x_2}\varphi \\ -\partial_{x_1}\partial_{x_3}\varphi + \partial_{x_3}\partial_{x_1}\varphi \\ \partial_{x_1}\partial_{x_2}\varphi - \partial_{x_2}\partial_{x_1}\varphi \end{pmatrix} \overset{\text{Clairaut/Schwarz}}{=} 0$$

- $$\operatorname{div}\operatorname{curl}\vec{f} = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix} \cdot \begin{pmatrix} \partial_{x_2}f_3 - \partial_{x_3}f_2 \\ -\partial_{x_1}f_3 + \partial_{x_3}f_1 \\ \partial_{x_1}f_2 - \partial_{x_2}f_1 \end{pmatrix}$$

$$= \partial_{x_1}(\partial_{x_2}f_3 - \partial_{x_3}f_2) + \partial_{x_2}(-\partial_{x_1}f_3 + \partial_{x_3}f_1) + \partial_{x_3}(\partial_{x_1}f_2 - \partial_{x_2}f_1) \underset{\text{Clairaut/Schwarz}}{=} 0$$

More examples in the homework/moodle exercises.

3. Ordinary Differential Equations

3.1 Basic Introduction

In many applications, we know relations between functions and their derivatives, e.g.,

- Newton's law: $m\frac{d^2\vec{x}(t)}{dt^2} = \underbrace{\vec{F}(\vec{x}(t))}_{\text{e.g., Coulomb interaction } \vec{F}(\vec{x}) = \text{const}\,\frac{\vec{x}}{|\vec{x}|^3}}$; given $\vec{F}$, we want to find the particle trajectory $\vec{x}(t)$
(different possibilities for different initial positions and velocities)

- Also different derivatives (and complex numbers) might be involved, as, e.g., in the Schrödinger equation (in quantum mechanics): $i\frac{\partial\psi(t,x)}{\partial t} = -\frac{\partial^2\psi(t,x)}{\partial x^2} + V(x)\psi(t,x)$
(1 dimensional equation for one particle; $\psi: \mathbb{R}^2 \to \mathbb{C}$)

- population growth: $\frac{dy(t)}{dt} = \lambda y(t)$ the more there is, the higher the increase ($\lambda > 0$ growth, $\lambda < 0$ decay)
Corona virus!

General setup:

Definition: For some given function f, we call

- $\underbrace{y'(x)}_{=\frac{dy(x)}{dx}} = f(x, y(x))$ a first order $\boxed{\text{ordinary differential equation (ODE)}}$
($y: \mathbb{R} \to \mathbb{R}$ and $f: \mathbb{R}^2 \to \mathbb{R}$ here).
If $y'(x) = f(y(x))$ (no explicit x-dependence) we say the ODE is $\boxed{\text{autonomous}}$.

- $\underbrace{y^{(n)}(x)}_{n\text{-th derivative}} = f(x, y(x), y'(x), \ldots, y^{(n-1)}(x))$ an $\boxed{n\text{-th order ODE}}$ ($y: \mathbb{R} \to \mathbb{R}$ and $f: \mathbb{R}^{n+1} \to \mathbb{R}$ here)

- $f(x_1, \ldots, x_n, y(\vec{x}), \underbrace{\frac{\partial y}{\partial x_1}, \ldots, \frac{\partial y}{\partial x_n}, \frac{\partial^2 y}{\partial x_1^2}, \frac{\partial^2 y}{\partial x_1 \partial x_2}, \ldots, \frac{\partial^n y}{\partial x_1 \cdots \partial x_1}}_{\text{all possible partial derivatives up to order } n}) = 0$ an n-th order $\boxed{\text{partial differential equation (PDE)}}$

Examples:
- population growth: 1st order ODE
- Newton's law: 2nd order ODE
- Schrödinger equation: 2nd order PDE

In this chapter we discuss some techniques to find solutions $y(x)$ for certain types of equations.
(Only ODEs in this chapter.)

Ex.: $\frac{dy}{dx} = \lambda y$, $y: \mathbb{R} \to \mathbb{R}, x \mapsto y(x)$, $\lambda \in \mathbb{R}$ fixed

formally, we can bring all y's and all x's to different sides: $\frac{dy}{y} = \lambda dx$, and then

integrate: $\int \frac{dy}{y} = \int \lambda dx \Rightarrow \ln y = \lambda x + C$

$\Rightarrow y(x) = e^{\lambda x + C} = e^C e^{\lambda x}$ is the solution!

this is called exponential growth for $\lambda > 0$ (or decay for $\lambda < 0$)

We actually have the freedom to choose C here. How is C determined? By the value of y at any x_0: $\underbrace{y(x_0)}_{=:y_0} = e^C e^{\lambda x_0}$, i.e., if x_0 and y_0 are given, we know C.

For some x_0, the $y(x_0)$ is called initial condition.

More clearly, we could just write our solution as $y(x) = \underbrace{y_0 e^{\lambda(x-x_0)}}_{\text{i.e., } e^C = y_0 e^{-\lambda x_0}}$ (s.t. y_0 is initial condition at x_0).

$y(x_0) = y_0 e^{\lambda(x_0 - x_0)} = y_0$

Often, one just chooses $x_0 = 0$, s.t. $y_0 = y(0)$.

Generally there is this important fact:

For an n-th order ODE $y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x))$, we need to specify n initial conditions $y(x_0), y'(x_0), \dots, y^{(n-1)}(x_0)$. In other words, the solution needs to have n independent constants.

Example: initial position $x(t_0)$ and initial velocity $x'(t_0)$ for Newton's equation $\frac{d^2\vec{x}(t)}{dt^2} = \vec{F}(\vec{x}(t))$

To summarize, the most important technique for solving ODEs is this:

Separation of variables:

this is certainly not always possible, e.g. $\frac{dy}{dx} = \cos(xy)$

For $\frac{dy}{dx} = f(y,x)$, we bring all x's to one side and all y's to the other (if possible), and then integrate both sides.

also here, we might or might not be able to actually perform the integration

3.2 Some Types of Integrable ODEs

integrable = find explicit solution by integration (Technique 1 above)

- $\boxed{y'(x) = f(x)g(y) \text{ is called separable ODE}}$

here, we can write $\frac{dy}{dx} = f(x)g(y)$, i.e., $\frac{dy}{g(y)} = f(x)dx$

$\Rightarrow$ we find the solution by integrating, $\int \frac{dy}{g(y)} = \int f(x)dx$ (if we can)

(note: at least we know that a solution exists if f and g are continuous and $g(y) \neq 0 \ \forall y$)

because then we can integrate

- $\boxed{y'(x) = f(x)\,y(x) \text{ is called linear homogeneous ODE}}$

as before: $\frac{dy}{dx} = f(x)y \Rightarrow \frac{dy}{y} = f(x)dx \Rightarrow \int \frac{dy}{y} = \int f(x)dx$

$\Rightarrow \ln y = \int f(x)dx + C \Rightarrow y(x) = e^{\int f(x)dx + C}$ (so we can always find a solution as long as f can be integrated)

- $\boxed{y'(x) = f(x)\,y(x) + g(x) \text{ is called linear inhomogeneous ODE}}$

here the idea is to write $y(x) = \underbrace{u(x)\,v(x)}_{\text{some product of two fct.s}}$ (s.t. applying the product rule gives a sum of two functions)

$$\Rightarrow y'(x) = (u(x)v(x))' = \frac{du}{dx}v + u\frac{dv}{dx} = f(x)\,u(x)v(x) + g(x)$$

$\Rightarrow$ solve first $\boxed{\frac{du}{dx} = f(x)\,u}$: $\frac{du}{u} = f(x)dx \overset{\text{as before}}{\Rightarrow} u(x) = e^{\int f(x)dx}$

next we need to solve $\boxed{u\frac{dv}{dx} = g(x)}$, i.e., $dv = \frac{g(x)}{u(x)}dx = e^{-\int f(x)dx}g(x)dx$

$$\Rightarrow v(x) = \int \underbrace{e^{-\int f(x)dx}}_{\text{this notation means this is a fct. of } x} g(x)dx + C$$

$$\Rightarrow \text{our solution is } y(x) = e^{\int f(x)dx}\left(\int e^{-\int f(x)dx} g(x)dx + C\right)$$

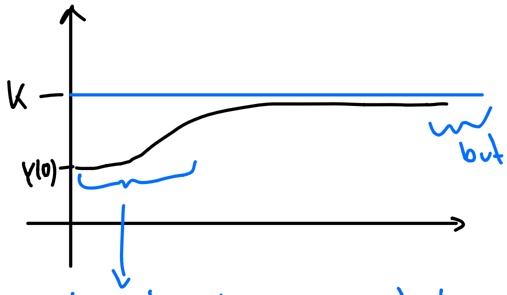
There are many examples, let us just give one (more in the exercises):

$\boxed{\text{Logistic growth}}$: $\frac{dy}{dx} = \underbrace{\lambda y}_{\text{this alone would lead to exponential growth}} \underbrace{\left(1 - \frac{y}{K}\right)}_{\text{growth is stopped once } y \text{ reaches } K}$ (λ: growth rate; K is sometimes called "environmental carrying capacity")

$\Rightarrow$ second order autonomous ODE

separation of variables: $\frac{dy}{\lambda y(1-\frac{y}{K})} = dx$

Integrating this will be a homework exercise. The result is: $y(x) = e^{\lambda x}\left(C + \frac{e^{\lambda x}}{K}\right)^{-1}$



at first, we have exponential growth

but for large x, $y(x) = \dfrac{e^{\lambda x}}{C + \frac{e^{\lambda x}}{K}} \approx \dfrac{e^{\lambda x}}{e^{\lambda x}/K} = K$