Session 16
March 30, 2020

We continue our discussion of the determinant.

Last time we established invariance of the determinant under the operation of adding a multiple of one row to another.

In Calculus and Linear Algebra I you learned that an $n \times n$ matrix has rank $< n$ if and only if after transforming it into upper triangular form (echelon form) there is at least one zero row at the bottom. But if that is the case, the determinant is zero. We have proven:

Theorem: Let $A \in \mathbb{R}^{n\times n}$ (a real $n\times n$ matrix). Then: • $\operatorname{rank} A < n \iff \det A = 0$

• $\operatorname{rank} A = n \iff \det A \neq 0$.

Keep the geometric reason in mind: $\operatorname{rank} A < n$ means the matrix maps $\vec{e}_1, \dots, \vec{e}_n$ into a lower dimensional subspace, so the area/volume etc. vanishes (e.g., a line has zero area, a plane has zero volume and so on).

Example from before: $\det \begin{pmatrix} 1 & 4 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} = -5 \neq 0$, so $\begin{pmatrix} 1 & 4 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$ has rank 3

(which we can also easily read off from the row vectors, which are clearly linearly independent).

Now $\operatorname{rank} A = n$ is the same as all row (or column) vectors being linearly independent. So:

Theorem: $\underbrace{\vec{R}_1, \vec{R}_2, \dots, \vec{R}_n \in \mathbb{R}^n \text{ are linearly independent}}_{\text{(Since we have } n \text{ vectors in } \mathbb{R}^n,\ \vec{R}_1, \dots, \vec{R}_n \text{ is a basis here.)}}$ $\overbrace{\text{iff}}^{\text{if and only if}}$ $\det \underbrace{\begin{pmatrix} \vec{R}_1 \\ \vdots \\ \vec{R}_n \end{pmatrix}}_{\text{matrix with rows } \vec{R}_1, \dots, \vec{R}_n} \neq 0.$

Two more properties:

D8) For $n\times n$ matrices A, and $\lambda \in \mathbb{R}$: $\det(\lambda A) = \lambda^n \det(A)$.

Geometric reason: If area/volume is scaled by λ in each direction $1,\dots,n$, the area/volume increases/decreases by a factor $\underbrace{\lambda\cdot\lambda\cdots\lambda}_{n \text{ times}}$.

Proof: Just use linearity (D1)) for each row.

D9) For $A, B \in \mathbb{R}^{n\times n}$, we have $\det(\underbrace{A\cdot B}_{\text{matrix product}}) = \det(A)\det(B)$.

(Here a geometric reason is not so intuitive at this point.)

(The proof is a bit lengthy, so let us not give it here, but one (slightly abstract) strategy is the following. Fix $A \in \mathbb{R}^{n\times n}$, then the map $B \mapsto \det(A\cdot B)$ satisfies properties D1)–D2), which actually already implies uniqueness up to the choice of the value at $\mathbb{1}$ (D3)). But here $\mathbb{1} \mapsto \det(A\cdot\mathbb{1}) = \det(A) = \det(A)\det(\mathbb{1})$, so $B \mapsto \det(AB) = \det(A)\det(B)$.)

Just to convince you: Take two upper triangular matrices A and B, i.e., for $i>j$ we have $A_{ij} = 0 = B_{ij}$. Then $(AB)_{ii} = \sum_j A_{ij}B_{ji} = A_{ii}B_{ii}$. So

$$\det(AB) = \det\begin{pmatrix} A_{11}B_{11} & & \text{sth.} \\ & \ddots & \\ 0 & & A_{nn}B_{nn}\end{pmatrix} = \prod_{i=1}^n A_{ii}B_{ii} = \det(A)\det(B).$$

For example:
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33}\end{pmatrix}\cdot\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33}\end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12}+a_{12}b_{22} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ 0 & a_{22}b_{22} & a_{22}b_{23}+a_{23}b_{33} \\ 0 & 0 & a_{33}b_{33}\end{pmatrix}$$

and the determinant is $(a_{11}b_{11})\cdot(a_{22}b_{22})\cdot(a_{33}b_{33}) = (a_{11}a_{22}a_{33})\cdot(b_{11}b_{22}b_{33})$.

Some consequences of D9):

- Let A be an invertible $n \times n$ matrix, i.e., $\exists\, A^{-1}$ s.t. $A^{-1}A = AA^{-1} = I$ (identity matrix). Then

$$1 = \det(I) = \det(A^{-1}A) = \det(A^{-1}) \cdot \det(A), \text{ so } \boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

- Generally: $\underbrace{\det(A^k)}_{= \det(\underbrace{A \cdot A \cdots A}_{k \text{ times}})} = \det(A)\det(A^{k-1}) = \ldots = (\det(A))^k$

But note: $\det(A+B) \neq \det(A) + \det(B)$ $\overbrace{\text{in general}}^{\text{It might hold for special choices of matrices, but not in general.}}$!

↳ Just take $A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}$. Then

$$\det(A+B) = \det\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ 0 & a_{22}+b_{22} \end{pmatrix} = (a_{11}+b_{11})(a_{22}+b_{22}), \text{ but}$$

$$\det(A) + \det(B) = \det\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} + \det\begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix} = a_{11}a_{22} + b_{11}b_{22} \neq \det(A+B) \text{ in general.}$$

A general note: Everything we have done so far for rows can also be done for columns.

In fact, we have the following:

D10) $\det(A) = \det(A^T)$, where A^T is the transpose of A (i.e. the matrix where rows and columns are interchanged: $(A^T)_{ij} = A_{ji}$).

"Geometric" intuition clear: It does not matter if we choose column or row vectors for computing the area/volume.

(A proof would again be a bit lengthy, so let us skip it here.)

Next: We already know the explicit formula for the determinant of a 2×2 matrix. Let us check it again with our properties:

↳ If $a_{11}\neq 0$: $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} = \det\begin{pmatrix} a_{11} & a_{12} \\ 0 & -\frac{a_{12}a_{21}}{a_{11}}+a_{22}\end{pmatrix} = a_{11}\left(-\frac{a_{12}a_{21}}{a_{11}}+a_{22}\right) = a_{11}a_{22}-a_{12}a_{21}.$ ✓

add $-\frac{a_{21}}{a_{11}}\cdot$ first row to second row ← note: if $a_{11}=0$, then

↳ If $a_{11}=0$: $\det\begin{pmatrix} 0 & a_{12} \\ a_{21} & a_{22}\end{pmatrix} \overset{D21}{=} -\det\begin{pmatrix} a_{21} & a_{22} \\ 0 & a_{12}\end{pmatrix} = -a_{21}a_{12} = 0\cdot a_{22} - a_{12}a_{21}.$ ✓

Graphically: $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}.$

or $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} = a_{11}\underbrace{\det\begin{pmatrix} \cancel{a_{11}} & \cancel{a_{12}} \\ \cancel{a_{21}} & a_{22}\end{pmatrix}}_{\text{meaning } \det a_{22}} - a_{12}\underbrace{\det\begin{pmatrix} \cancel{a_{11}} & \cancel{a_{12}} \\ a_{21} & \cancel{a_{22}}\end{pmatrix}}_{\text{meaning } \det a_{21}} = a_{11}a_{22}-a_{12}a_{21}.$ ✓

One should also know the explicit formula for 3×3 matrices. We could compute it similarly to above, but this is very lengthy. There is an easier way called Laplace expansion, which is inspired by what we did here. Let me state what it is first.

<u>Definition:</u> For $A\in\mathbb{R}^{n\times n}$ and any $i,j\in\{1,\dots,n\}$, we define the $\boxed{a_{ij} \text{ minor } \operatorname{mnr}(a_{ij})}$ as the determinant of the $(n-1)\times(n-1)$ matrix A with removing row i and column j.
We call $\operatorname{cof}(a_{ij}) = (-1)^{i+j}\operatorname{mnr}(a_{ij})$ the $\boxed{a_{ij}\text{-cofactor}}$.

E.g., for $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix}$ we have $\operatorname{mnr}(a_{11}) = \det\begin{pmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33}\end{pmatrix} = \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{pmatrix} \overset{\text{since } (-1)^{1+1}=1}{=} \operatorname{cof}(a_{11}),$

and $\operatorname{cof}(a_{12}) = (-1)^{1+2} \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = -\det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$,

and $\operatorname{cof}(a_{13}) = (-1)^{1+3} \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$.

These definitions are so useful because we have the following theorem:

<u>Theorem:</u> The determinant of any $n \times n$ matrix A is equal to its Laplace expansion

by any row, i.e., $\det(A) = \sum_{j=1}^{n} a_{ij} \operatorname{cof}(a_{ij})$ for any $i = 1, \dots, n$, or

by any column, i.e., $\det(A) = \sum_{i=1}^{n} a_{ij} \operatorname{cof}(a_{ij})$ for any $j = 1, \dots, n$.

What does this mean, e.g., for 3×3 matrices?

Let us do a Laplace expansion by the first row:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= (-1)^{1+1} a_{11} \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + (-1)^{1+2} a_{12} \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + (-1)^{1+3} a_{13} \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$$= a_{11} \left(a_{22} a_{33} - a_{23} a_{32} \right) - a_{12} \left(a_{21} a_{33} - a_{23} a_{31} \right) + a_{13} \left(a_{21} a_{32} - a_{22} a_{31} \right).$$

$$= a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33}.$$

Note: One can remember this last expression with the rule of Sarrus:

add up the product of the encircled numbers

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ "=" } \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

meaning: $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$ | meaning: $a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$

So the Laplace expansion is indeed true for 2×2 matrices (and we could check it explicitly for 3×3 matrices, too.)

The general case can be proven, e.g., by induction (using D5)) or by checking that the expansion also satisfies D1)–D3) and then use the uniqueness. But let us skip the proof here.

Note:
- One can use any row or any column for the Laplace expansion.
- E.g., we could compute determinants of 4×4 matrices by repeated Laplace expansion.
- If a row or column has one or more zeros, one should do the Laplace expansion using that row or column.

Example:

$$\det\begin{pmatrix} 1 & 3 & 4 & 1 \\ 3 & 0 & 2 & 0 \\ 2 & 0 & 1 & 4 \\ 1 & 0 & 2 & 3 \end{pmatrix} \overset{\substack{\text{Laplace expansion} \\ \text{by 2nd column}}}{=} (-1)^{1+2}\, 3 \det\begin{pmatrix} 3 & 2 & 0 \\ 2 & 1 & 4 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\overset{\substack{\text{Laplace expansion} \\ \text{by first row}}}{=} -3\cdot\left(3\cdot\det\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - 2\det\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}\right)$$

$$= -3\left(3\cdot(3-8) - 2\cdot(6-4)\right)$$

$$= -3(-15-4) = 57$$