Session 17
April 1, 2020

Summary: What do we know about determinants so far?

- Geometric intuition: $\det\begin{pmatrix}\vec{R}_1\\ \vdots \\ \vec{R}_n\end{pmatrix}$ is the volume of the parallelepiped spanned by the row vectors $\vec{R}_1, \dots, \vec{R}_n$.
- The map $\det: \mathbb{R}^{n\times n} \to \mathbb{R}$ with properties D1), D2), D3) is unique.
- det has properties D1) – D10).
- We can compute $\det A$ with Gaussian elimination.
- The connection of det to rank and linear independence is very useful.
- $\det A$ can be computed with the Laplace expansion (often more convenient than Gaussian elimination).
- Remember the explicit formulas for the determinant for 2×2 and 3×3 matrices!

Today: A few more things you can do with determinants.

Cramer's rule:

Let us consider a system of 3 equations with 3 unknowns:

$A\vec{x} = \vec{b}$, where $A \in \mathbb{R}^{3\times 3}$, $\vec{x}, \vec{b} \in \mathbb{R}^3$, i.e., explicitly:

$$A_{11}x_1 + A_{12}x_2 + A_{13}x_3 = b_1$$
$$A_{21}x_1 + A_{22}x_2 + A_{23}x_3 = b_2$$
$$A_{31}x_1 + A_{32}x_2 + A_{33}x_3 = b_3$$

Let us compute $\det A$ in a special way:

$$\det A = \det\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \overset{\text{D5) and D10)}}{=} \det\begin{pmatrix} \underbrace{A_{11} + \frac{x_2}{x_1}A_{12} + \frac{x_3}{x_1}A_{13}}_{=\frac{b_1}{x_1}} & A_{12} & A_{13} \\ \underbrace{A_{21} + \frac{x_2}{x_1}A_{22} + \frac{x_3}{x_1}A_{23}}_{=\frac{b_2}{x_1}} & A_{22} & A_{23} \\ \underbrace{A_{31} + \frac{x_2}{x_1}A_{32} + \frac{x_3}{x_1}A_{33}}_{=\frac{b_3}{x_1}} & A_{32} & A_{33} \end{pmatrix}$$

add $\frac{x_2}{x_1}\cdot$(second column) $+ \frac{x_3}{x_1}\cdot$(third column) to first column

$$= \det\begin{pmatrix} b_1/x_1 & A_{12} & A_{13} \\ b_2/x_1 & A_{22} & A_{23} \\ b_3/x_1 & A_{32} & A_{33} \end{pmatrix} \overset{\text{D1) and D10)}}{=} \frac{1}{x_1}\det\begin{pmatrix} b_1 & A_{12} & A_{13} \\ b_2 & A_{22} & A_{23} \\ b_3 & A_{32} & A_{33} \end{pmatrix}$$

So we have found that
$$x_1 = \frac{\det\begin{pmatrix} b_1 & A_{12} & A_{13} \\ b_2 & A_{22} & A_{23} \\ b_3 & A_{32} & A_{33} \end{pmatrix}}{\det\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}} =: \frac{\det A_{1\to\vec{b}}}{\det A}$$

matrix where first column is replaced by $\vec{b}$

analogously: $x_2 = \dfrac{\det A_{2\to\vec{b}}}{\det A}$, $x_3 = \dfrac{\det A_{3\to\vec{b}}}{\det A}$

Of course, this only works when $\det A \neq 0$. When $\det A = 0$ we already know that $\operatorname{rank} A < n$, i.e., there is no <u>unique</u> solution (there are ∞-many or none).

The general result is

$$\boxed{\begin{array}{l}\textbf{Theorem (Cramer's Rule): } \text{A square system of } n \text{ linear equations } A\vec{x} = \vec{b} \; (A \in \mathbb{R}^{n\times n}, \\ \vec{x}, \vec{b} \in \mathbb{R}^n) \text{ has a unique solution if and only if } \det A \neq 0. \text{ In this case the explicit} \\ \text{solution is } x_j = \dfrac{\det A_{j\to\vec{b}}}{\det A}, \text{ where } A_{j\to\vec{b}} \text{ is the matrix with the } j\text{-th column replaced by } \vec{b}.\end{array}}$$

Note: • For a $\boxed{\text{homogeneous system } A\vec{x}=0 \text{ this means: Only } \vec{x}=0 \text{ is a solution} \iff \det A \neq 0.}$

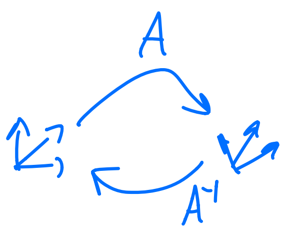
• Cramer's rule is a very nice theoretical result and very useful if only one x_j is needed (or a few). Otherwise Gaussian elimination is more practical.

Ex.: Solve
$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 1 \\ 5x_1 + 3x_2 + 5x_3 &= 2 \\ 6x_1 + 4x_2 + 4x_3 &= 3 \end{aligned} \qquad \text{for } x_3.$$

First: $$\det\begin{pmatrix} 1 & 2 & 2 \\ 5 & 3 & 5 \\ 6 & 4 & 4 \end{pmatrix} \overset{\text{Sarrus}}{=} 1\cdot3\cdot4 + 2\cdot5\cdot6 + 2\cdot5\cdot4 - 2\cdot3\cdot6 - 5\cdot4\cdot1 - 4\cdot2\cdot5$$
$$= 12 + 60 + 40 - 36 - 20 - 40$$
$$= 16 \implies \text{unique solution for any } \vec{b}, \text{ in particular } \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\implies x_3 = \frac{\det\begin{pmatrix} 1 & 2 & 1 \\ 5 & 3 & 2 \\ 6 & 4 & 3 \end{pmatrix}}{16} = \frac{1}{16}\left(1\cdot3\cdot3 + 2\cdot2\cdot6 + 1\cdot5\cdot4 - 1\cdot3\cdot6 - 2\cdot4\cdot1 - 3\cdot2\cdot5\right)$$
$$= \frac{1}{16}\left(9 + 24 + 20 - 18 - 8 - 30\right)$$
$$= -\frac{3}{16}$$

Matrix Inverse:



Recall:

• A square matrix $A \in \mathbb{R}^{n\times n}$ is called invertible (or non-singular) if there is a square matrix $A^{-1} \in \mathbb{R}^{n\times n}$ such that $A^{-1}A = AA^{-1} = \underbrace{I_{n\times n}}_{=\,n\times n \text{ identity matrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}$.

- In Calculus and Linear Algebra I you learned how to compute A^{-1} with Gaussian elimination.

- If A is invertible, the linear system $A\vec{x}=\vec{b}$ has the unique solution $\vec{x}=A^{-1}\vec{b}$
(In particular, $A\vec{x}=\vec{0}$ has only the solution $\vec{x}=A^{-1}\vec{0}=\vec{0}$.)

- $\boxed{A^{-1} \text{ exists} \iff \operatorname{rank} A = n}$

The last point shows that a (square) matrix is invertible if and only if $\det A \neq 0$.
If $\det A = 0$, the matrix A is not invertible and called singular.

We can even give an explicit formula for the inverse (if it exists) using determinants and the Laplace expansion. For that, we define the following:

Definition: For an $n\times n$ matrix A we define an $n\times n$ matrix $\operatorname{Adj} A$, called the classical adjoint of A by $(\operatorname{Adj} A)_{ij} = \underbrace{\operatorname{cof} A_{ji}}$.

cofactor $:= (-1)^{i+j}\det(A$ with removing row j and column $i)$

In other words, if C is the matrix of cofactors ($C_{ij}=\operatorname{cof} A_{ij}$), then $\operatorname{Adj} A = \underbrace{C^T}$.

C transpose, i.e., rows and columns interchanged

Then we have:

Theorem: Let $A\in\mathbb{R}^{n\times n}$. Then A is invertible if and only if $\det A\neq 0$, and in this case $A^{-1}=\frac{1}{\det A}\operatorname{Adj} A$.

A proof can be given using the Laplace expansion (see RHB Ch. 8.10), but let us skip it here.

Ex.: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 2 \end{pmatrix}$.

We find $\det A = 1\cdot5\cdot2 + 2\cdot6\cdot3 + 3\cdot4\cdot2 - 3\cdot5\cdot3 - 6\cdot2\cdot1 - 2\cdot2\cdot4$

$= 10 + 36 + 24 - 45 - 12 - 16$

$= -3 \quad \Rightarrow A$ is invertible

Furthermore: $\operatorname{cof} A_{11} = \det\begin{pmatrix} 5 & 6 \\ 2 & 2 \end{pmatrix} = 10 - 12 = -2$

$\operatorname{cof} A_{12} = -\det\begin{pmatrix} 4 & 6 \\ 3 & 2 \end{pmatrix} = -(8 - 18) = 10$

$\operatorname{cof} A_{13} = \det\begin{pmatrix} 4 & 5 \\ 3 & 2 \end{pmatrix} = 8 - 15 = -7$

$\operatorname{cof} A_{21} = -\det\begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix} = -(4 - 6) = 2$

$\operatorname{cof} A_{22} = \det\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} = 2 - 9 = -7$

$\operatorname{cof} A_{23} = -\det\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} = -(2 - 6) = 4$

$\operatorname{cof} A_{31} = \det\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} = 12 - 15 = -3$

$\operatorname{cof} A_{32} = -\det\begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix} = -(6 - 12) = 6$

$\operatorname{cof} A_{33} = \det\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} = 5 - 8 = -3$

$\Rightarrow$ cofactor matrix $C = \begin{pmatrix} -2 & 10 & -7 \\ 2 & -7 & 4 \\ -3 & 6 & -3 \end{pmatrix}$

$\Rightarrow$ classical adjoint $\operatorname{Adj} A = C^T = \begin{pmatrix} -2 & 2 & -3 \\ 10 & -7 & 6 \\ -7 & 4 & -3 \end{pmatrix}$

$$\Longrightarrow \text{inverse } A^{-1} = -\frac{1}{3}\begin{pmatrix} -2 & 2 & -3 \\ 10 & -7 & 6 \\ -7 & 4 & -3 \end{pmatrix}$$

Consistency check: $$A^{-1}A = -\frac{1}{3}\begin{pmatrix} -2 & 2 & -3 \\ 10 & -7 & 6 \\ -7 & 4 & -3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 2 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} -2+8-9 & -4+10-6 & -6+12-6 \\ 10-28+18 & 20-35+12 & 30-42+12 \\ -7+16-9 & -14+20-6 & -21+24-6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \checkmark$$

Finally, I should make one more remark on determinants (just if you are interested, not relevant for exam.)

We have tiptoed around a clear direct definition of the determinant, and have instead talked about the intuition behind it and its properties, which I thought was more insightful. We claimed that the determinant is the unique map that satisfies properties D1), D2), D3). But one can also give an explicit expression for this map.

We need the following definition:

<u>Definition:</u> A bijection from $\{1,2,\dots,n\}$ to itself is called permutation of the numbers $1,\dots,n$. The set of all permutations of $1,\dots,n$ is called S_n (the symmetric group).

In other words, permutations are just rearrangements of the numbers $1,\dots,n$.

Ex.: $\sigma: \{1,2,3\} \to \{1,2,3\}$, def. by $\sigma(1)=2$, $\sigma(2)=3$, $\sigma(3)=1$, i.e.:

$$\begin{array}{c|c|c|c} & 1 & 2 & 3 \\ \hline \sigma & 2 & 3 & 1 \\ & \uparrow & \uparrow & \uparrow \\ & \sigma(1) & \sigma(2) & \sigma(3) \end{array}$$

E.g., for $n=3$ there are six possible permutations:

σ	1	2	3
	1	2	3
	3	1	2
	2	3	1
	3	2	1
	1	3	2
	2	1	3

All permutations can be put together from interchanging pairs:

e.g., we can get to $\sigma(1)=2, \sigma(2)=3, \sigma(3)=1$ by going $1,2,3 \to 2,1,3 \to 2,3,1$

If we need an even number of such pair interchanges, we say the permutation σ is even and its sign is $+1$ or $\operatorname{sgn}\sigma = +1$; for odd we say its sign is -1 or $\operatorname{sgn}\sigma = -1$.

Ex.: $\sigma(1)=2, \sigma(2)=3, \sigma(3)=1$ is even, since we need 2 interchanges

Now one can show that the following map satisfies D1), D2), D3):

<u>Theorem (Leibniz formula):</u> $$\det A = \underbrace{\sum_{\sigma\in S_n}}_{\substack{\text{sum over all}\\ \text{possible}\\ \text{permutations}}} \underbrace{(\operatorname{sgn}\sigma)}_{\substack{\text{sign of the}\\ \text{permutation}\\ (+1 \text{ or } -1)}} \cdot A_{1\sigma(1)} \cdot A_{2\sigma(2)} \cdots \underbrace{A_{n\sigma(n)}}_{\substack{\text{the } n,\sigma(n) \text{ entry}\\ \text{of the matrix } A}} .$$

Ex.: For $n=2$ there is the even permutation $\sigma(1)=1, \sigma(2)=2$ (the identity) and the odd permutation $\sigma(1)=2, \sigma(2)=1$.

So $\det\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix} = (+1)\cdot A_{11}\cdot A_{22} + (-1)\, A_{12}\, A_{21}$ (coincides with what we know)

If you like, check that this coincides with our previous computations for $n=3$ also.

Summary: Methods to compute determinants

- Bring matrix into upper triangular form by Gaussian elimination, then det is the product of the diagonal entries.

- Repeated Laplace expansion.

- Explicit formulas for 2×2 matrices and 3×3 matrices (Sarrus)

(• Leibniz formula)