Session 18
April 15, 2020

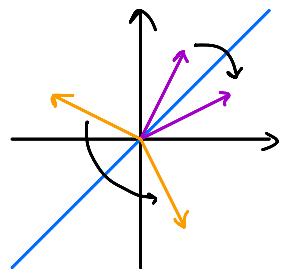
note: take a look at the new HW sheet 5 for the role of determinants in many-dimensional integration

last few lectures: we have introduced the determinant extensively and have already seen some nice applications (linear independence, inverses, systems of linear equations). For the next few classes we will use determinants again every now and then, especially today.

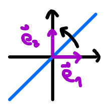
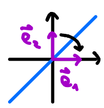
4.3 Eigenvalues and Eigenvectors

Let us consider the action of a matrix – or generally, a linear operator – on vectors again.

Take the example of a reflection across the diagonal:



What is the corresponding matrix? Let us check how it acts on the basis vectors:

- $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is mapped to $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ 
- $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is mapped to $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ 

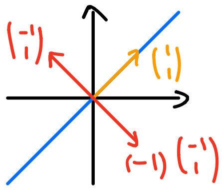
so, of course, the matrix just interchanges x and y

$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$

So the matrix should have $\vec{e}_2$ as first column and $\vec{e}_1$ as second: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Now, note that there are a few vectors on which the matrix acts in a very simple way, namely that it doesn't change the vector expect for a scalar multiple. These are $\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (let's choose them normalized), because

$A\vec{v}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \vec{v}_1$ and $A\vec{v}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\vec{v}_2$



So wouldn't it be better if we write our vectors in the basis $\mathcal{B}' = \{\vec{v}_1, \vec{v}_2\}$ instead of $\mathcal{B} = \{\vec{e}_1, \vec{e}_2\}$? Then the form of the matrix would be very simple.

Let's do this: Above we have computed that the map in the basis $\mathcal{B}'$ has $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as first, and $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ as second column vector: $A_{\mathcal{B}'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(Double check: $A_{\mathcal{B}'} \underbrace{(\vec{v}_1)_{\mathcal{B}'}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\mathcal{B}'} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{B}'} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{B}'} \checkmark$ $\quad A_{\mathcal{B}'}(\vec{v}_2)_{\mathcal{B}'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\mathcal{B}'} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathcal{B}'} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}_{\mathcal{B}'} = -\vec{v}_2 \checkmark$)

$\vec{v}_1$ in the basis $\mathcal{B}'$ is just $1\cdot\vec{v}_1 + 0\cdot\vec{v}_2$, so $(\vec{v}_1)_{\mathcal{B}'} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

[recall: if $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis, we can write any vector $\vec{v}$ as $\vec{v} = \sum_{i=1}^{n} c_i \vec{v}_i$ for some unique coefficients c_i, $i = 1, \dots, n$.
We commonly write this as $\vec{v} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_{\mathcal{B}}$. If the basis is the standard basis $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \vec{e}_n$, we leave the subscript $\mathcal{B}$ away.]

(Side note: Please recall from Calculus and Linear Algebra I how to write matrices and vectors in different bases.)

Conclusion $A_{\mathcal{B}'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This is such a simple matrix, because by construction it is diagonal. (And those are the nicest possible type of matrices.)

Today and later, we explore this idea further for general matrices, e.g.,

- How do we find vectors such as $\vec{v}_1, \vec{v}_2$ for arbitrary matrices A?
- Can we always find a basis in which a given matrix is diagonal?
- What can we use it for? (See video on class website for an "application" ☺)

Let us introduce some names first:

Definition: Let A be a real $n \times n$ matrix. If $\vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$ satisfies $\boxed{A\vec{x} = \lambda \vec{x}}$ for some $\lambda \in \mathbb{R}$, we call λ an **eigenvalue** of A, and $\vec{x}$ an **eigenvector** (corresponding to λ).

Note: • "Eigen" is a German word ☺

• The eigenvectors are only determined up to a scalar multiple: If $\vec{x}$ is an eigenvector corresponding to eigenvalue λ, then also $A(c\vec{x}) = cA\vec{x} = c\lambda\vec{x} = \lambda(c\vec{x})$ $\forall c \in \mathbb{R}$, i.e., also $c\vec{x}$ for any $c \neq 0$ is an eigenvector. Usually it is most convenient to take eigenvectors normalized (but there might be linearly independent eigenvectors for the same eigenvalue; we will discuss this more detailed later).

• Eigenvalues and eigenvectors will come up again in ALL of your majors!

Now, how do we find eigenvalues and eigenvectors for a given matrix A?

We need to satisfy $A\vec{x} = \lambda\vec{x}$, i.e., $A\vec{x} - \lambda\vec{x} = \vec{0}$, i.e., $(A - \lambda \overset{\text{identity matrix}}{I})\vec{x} = 0$.

This is just a system of linear equations! It has the unique solution $\vec{x} = 0$ if and only if $\det(A - \lambda I) \neq 0$. But we are looking for the opposite, namely <u>non-zero</u> solutions $\vec{x}$.

So we want $\underbrace{\det(A - \lambda I)}_{\text{This is some polynomial in } \lambda \text{ (of degree } n\text{), and we need to find it's zeroes.}} = 0$.

$\Leftrightarrow$ $A - \lambda I$ singular $\Leftrightarrow$ rank$(A-\lambda I) < n$ $\Leftrightarrow$ nullity$(A - \lambda I) \neq 0$
$\Leftrightarrow$ rows linearly dependent $\Leftrightarrow$ columns linearly dependent

For example, in the simple case $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ above, we find

$$A - \lambda I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}, \text{ and } \det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1$$

This is zero for $\lambda = \pm 1$, as we found above.

Note: We know how to find zeroes of polynomials of degree 2, but for degrees 3 or higher this can become very hard or even impossible in general.

To summarize:

Definition: For $A \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{R}$, we call $P(\lambda) = \det(A - \lambda I)$ the characteristic polynomial of A. The equation $\det(A-\lambda I) = 0$ is called characteristic equation.

Theorem: The eigenvalues λ of $A \in \mathbb{R}^{n\times n}$ are the zeroes of its characteristic polynomial:

$$\det(A - \lambda I) = 0.$$

Ex.: Find the eigenvalues of $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 6 \\ 0 & 1 & 2 \end{pmatrix}$.

We compute the characteristic polynomial:

$$P(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & 1-\lambda & 6 \\ 0 & 1 & 2-\lambda \end{pmatrix} = -\lambda(1-\lambda)(2-\lambda) + 6\lambda = -\lambda^3 + 3\lambda^2 + 4\lambda$$

Now find the zeroes: $0 = -\lambda^3 + 3\lambda^2 + 4\lambda = -\lambda(\lambda^2 - 3\lambda - 4)$

$$\Rightarrow \lambda_0 = 0 \text{ and } \lambda_\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} + 4} = \frac{3}{2} \pm \frac{5}{2} \Rightarrow \lambda_+ = 4,\ \lambda_- = -1.$$

The eigenvalues are 0, 4 and -1.

What are the corresponding eigenvectors? We need to compute them separately for each eigenvalue:

$$(A - \lambda_0 I)\vec{x}_0 = \vec{0} \Rightarrow \left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 6 \\ 0 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{aligned} y_0 \quad\quad &= 0 \\ y_0 + 6z_0 &= 0 \\ y_0 + 2z_0 &= 0 \end{aligned} \Rightarrow y_0 = 0, z_0 = 0 \quad (x_0 \text{ free})$$

$$\Rightarrow \vec{x}_0 = \begin{pmatrix} x_0 \\ 0 \\ 0 \end{pmatrix} \text{ for any } x_0 \neq 0 \text{ are all the normalized eigenvectors corresponding to } \lambda_0.$$

$\vec{x}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is the only normalized eigenvector corresponding to λ_0.

The other two:

$$(A - \lambda_+ I)\vec{x}_+ = \vec{0} \Rightarrow \begin{pmatrix} -4 & 1 & 0 \\ 0 & 1-4 & 6 \\ 0 & 1 & 2-4 \end{pmatrix} \vec{x}_+ = \vec{0} = \begin{pmatrix} -4 & 1 & 0 \\ 0 & -3 & 6 \\ 0 & 1 & -2 \end{pmatrix} \vec{x}_+$$

$$\Rightarrow \begin{array}{l} -4x_+ + y_+ = 0 \\ -3y_+ + 6z_+ = 0 \\ y_+ - 2z_+ = 0 \end{array}$$

multiplying last row by (-3), we see that both equations are the same (as at least two of them should be, otherwise $\vec{0}$ would be the unique solution, meaning we would have made a mistake).

$$\Rightarrow y_+ = 2z_+ \overset{\text{plug into 1st row}}{\Rightarrow} -4x_+ + 2z_+ = 0 \Rightarrow x_+ = \tfrac{1}{2} z_+$$

now we can choose, e.g., z_+ freely, and, if desired, determine it by normalization:

$\vec{x}_+ = \begin{pmatrix} \frac{1}{2} z_+ \\ 2z_+ \\ z_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 2 \\ 1 \end{pmatrix} z_+$ for any $z_+ \neq 0$ are all eigenvectors corresponding to $\lambda_+ = 4$.

$$|\vec{x}_+| = \sqrt{\tfrac{1}{4} + 4 + 1}\, z_+ = \frac{\sqrt{21}}{2} z_+ \overset{!}{=} 1 \Rightarrow z_+ = \frac{2}{\sqrt{21}}$$

$\Rightarrow \vec{x}_+ = \frac{2}{\sqrt{21}} \begin{pmatrix} \frac{1}{2} \\ 2 \\ 1 \end{pmatrix}$ is the normalized eigenvector

For λ_- one finds by a similar computation the corresponding normalized eigenvector

$$\vec{x}_- = \frac{1}{\sqrt{19}} \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}.$$

Summary: $A=\begin{pmatrix}0&1&0\\0&1&6\\0&1&2\end{pmatrix}$ has three eigenvalues $\lambda_0=0$, $\lambda_+=4$, $\lambda_-=-1$ and

corresponding normalized eigenvectors $\vec{x}_0=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\vec{x}_+=\frac{2}{\sqrt{21}}\begin{pmatrix}\frac{1}{2}\\2\\1\end{pmatrix}$, $\vec{x}_-=\frac{1}{\sqrt{19}}\begin{pmatrix}3\\-3\\1\end{pmatrix}$.

As you can see, the computations can become lengthy (though not hard) already for 3x3 matrices. Luckily, we can always easily test if our computations were correct:

$$A\vec{x}_0=\begin{pmatrix}0&1&0\\0&1&6\\0&1&2\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\vec{0} \quad \checkmark$$

$$A\vec{x}_+=\begin{pmatrix}0&1&0\\0&1&6\\0&1&2\end{pmatrix}\frac{2}{\sqrt{21}}\begin{pmatrix}\frac{1}{2}\\2\\1\end{pmatrix}=\frac{2}{\sqrt{21}}\begin{pmatrix}2\\2+6\\2+2\end{pmatrix}=4\cdot\frac{2}{\sqrt{21}}\begin{pmatrix}\frac{1}{2}\\2\\1\end{pmatrix}=\lambda_+\vec{x}_+ \quad \checkmark$$

$$A\vec{x}_-=\begin{pmatrix}0&1&0\\0&1&6\\0&1&2\end{pmatrix}\frac{1}{\sqrt{19}}\begin{pmatrix}3\\-3\\1\end{pmatrix}=\frac{1}{\sqrt{19}}\begin{pmatrix}-3\\-3+6\\-3+2\end{pmatrix}=-1\cdot\frac{1}{\sqrt{19}}\begin{pmatrix}3\\-3\\1\end{pmatrix}=\lambda_-\vec{x}_- \quad \checkmark$$

<u>Summary:</u> How to compute eigenvalues and vectors of an nxn matrix A?

- Find the characteristic polynomial $\det(A-\lambda I)$.
- The eigenvalues are the solution to the characteristic equation $\det(A-\lambda I)=0$.
- For each eigenvalue, solve the system of linear equations $(A-\lambda I)\vec{x}_\lambda=\vec{0}$; all such non-zero $\vec{x}_\lambda$ are eigenvectors.

Next, we will study eigenvalues and eigenvectors more systematically, e.g.:

- Do they always exist?
- How many are there?
- Things we can do with them