Session 25
May 11, 2020

4.7.3 Singular Value Decomposition

Finally, let us briefly discuss one of the most important matrix decompositions, both for theory and applications. It works for any $m\times n$ real or complex matrix. Let us here just state the decomposition, and then find out how to compute it.

Theorem:

Any $m\times n$ matrix A has a singular value decomposition (SVD) $\boxed{A = U\Sigma V^\dagger}$, where

- U is a unitary $m\times m$ matrix,
- V is a unitary $n\times n$ matrix, and
- Σ is an $m\times n$ matrix with only non-negative values σ_i (i.e., $\sigma_i \geq 0$) on the diagonal, and 0's everywhere else.

(i.e., $\Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ 0 & \ddots & \sigma_n \\ 0 & \cdots & 0 \end{pmatrix}$ for $m>n$, $\Sigma = \begin{pmatrix} \sigma_1 & & 0 & 0 & \cdots & 0 \\ 0 & \ddots & \sigma_m & 0 & \cdots & 0 \end{pmatrix}$ for $m<n$,

$\Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ 0 & \ddots & \sigma_n \end{pmatrix}$ for $m=n$.)

The σ_i are called singular values of A, and they are uniquely determined by A.

Now, let us assume $A = U\Sigma V^\dagger$ as above, and let us see how to find first Σ, and then U and V.

recall: $(AB)^\dagger = B^\dagger A^\dagger$

We have $A^\dagger A = (U\Sigma V^\dagger)^\dagger (U\Sigma V^\dagger) = \underbrace{(V^\dagger)^\dagger}_{=V} \Sigma^\dagger \underbrace{U^\dagger U}_{=I \text{, since } U \text{ unitary}} \Sigma V^\dagger = V\Sigma^\dagger \Sigma V^\dagger$,

and similarly $AA^\dagger = (U\Sigma V^\dagger)(U\Sigma V^\dagger)^\dagger = U\Sigma V^\dagger (V^\dagger)^\dagger \Sigma^\dagger U^\dagger = U\Sigma\Sigma^\dagger U^\dagger$.

Here, $\Sigma^+\Sigma$ is an $n\times n$ matrix with $\sigma_1^2,\ldots,\sigma_k^2$ on the diagonal $(k=\min(m,n))$, and $\Sigma\Sigma^+$ is an $m\times m$ matrix with $\sigma_1^2,\ldots,\sigma_k^2$ on the diagonal; all other matrix entries are 0.

And A^+A and AA^+ are both Hermitian! So they can be diagonalized with unitary matrices. The two equations above are exactly their diagonalization:

- $A^+A = V\Sigma^+\Sigma V^+$
- $AA^+ = U\Sigma\Sigma^+U^+$.

$\Rightarrow$ $\sigma_1^2,\ldots,\sigma_k^2$ are the eigenvalues of both A^+A and AA^+; the larger of those two matrices has additionally $|m-n|$ eigenvalues 0.

$\Rightarrow$ The orthonormal eigenvectors of A^+A are the columns of V.

$\Rightarrow$ The orthonormal eigenvectors of AA^+ are the columns of U.

This is how the singular value decomposition can be computed.

Examples are a bit lengthy, since we have to diagonalize two matrices. Some simple examples are in the moodle exercises.

But one can easily check whether a given decomposition is an SVD: e.g.,

$$\underbrace{\begin{pmatrix} 3 & 3 \\ -1 & 1 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{V^+}$$

holds, and U and V are unitary,

and Σ has positive values on the diagonal.

Thank you for a nice semester, good luck with all of your exams, and I'm looking forward too seeing you around and in other classes in the following semesters!

Don't forget to fill out the teaching evaluations!