Moodle Exercise Set 7

Calculus and Linear Algebra II

Spring 2020

1. Find the derivative of the vector-valued function $r(t) = (t-2, t^2+1)$ at $t = -1$.
2. Find the derivative of the vector-valued function $r(t) = (\cos(t), 3t, 2\sin(2t))$ at $t = 0$.
3. A vector-valued function $r : \mathbb{R} \to \mathbb{R}^3$ describes a curve in $\mathbb{R}^3$. The length of this curve between the points $r(a)$ and $r(b)$ is given by the integral $\int_a^b \sqrt{r'(t) \cdot r'(t)}dt$. Find the length of the helix defined by $r(t) = (\cos(t), \sin(t), t)$ from the point $(1, 0, 0)$ to $(1, 0, 2\pi)$.
4. For a vector-valued function $r : \mathbb{R} \to \mathbb{R}^2$, the function describing the unit tangent vector at any point is given by $T(t) = \frac{r'(t)}{\sqrt{r'(t) \cdot r'(t)}}$. The curvature of the curve at any point is defined by $\kappa(t) = \sqrt{\frac{T'(t) \cdot T'(t)}{r'(t) \cdot r'(t)}}$. Consider the function $r(t) = (a\cos(t), a\sin(t))$describing a circle of radius a around the origin. What is the curvature of the circle?
5. Recall that a if vector-valued function $r(t)$ describes the position of a particle, then $v(t) = r'(t)$ is its velocity and $a(t) = v'(t)$ is its acceleration. If the acceleration (resp. velocity) is known, the velocity (resp. position) can be computed by integration. A moving particle starts at an initial position $r(0) = (1, 0, 0)$ with velocity $v(0) = (1, -1, 1)$. If its acceleration is $a(t) = (4t, 6t, 1)$. Find the position of the particle at $t = 1$.
6. Consider the function $f(r, \theta) = (e^{-r}\sin(\theta), e^r\cos(\theta))$. Compute the Jacobian matrix of f.
7. Find the curl and the divergence of $F(x, y, z) = (xyz, 0, -x^2y)$.
8. Find the curl and the divergence of $F(x, y, z) = (1, x + yz, xy - \sqrt{z})$.
9. Find the curl and the divergence of $F(x, y, z) = (\ln(x), \ln(xy), \ln(xyz))$.
10. Maxwell's equations relating the electric field E and magnetic field H as they vary in time in a region containing no charge and no current can be stated as follows

$$\begin{aligned} \operatorname{div}(E) &= 0 & \qquad \operatorname{div}(H) &= 0 \\ \operatorname{curl}(E) &= -\frac{1}{c}\frac{\partial H}{\partial t} & \qquad \operatorname{curl}(H) &= \frac{1}{c}\frac{\partial E}{\partial t} \end{aligned}$$

    where c is the speed of light. Compute $\nabla \times (\nabla \times E)$ in terms of E.
11. Solve the differential equation $(x^2 + 1)y' = xy$.
12. Solve the differential equation $\frac{dy}{dt} = \frac{te^t}{y\sqrt{1+y^2}}$
13. Solve the differential equation $\frac{du}{dt} = \frac{2t+\sec^2(t)}{2u}$ with the initial condition $u(0) = -5$ (recall that $\sec(\theta) = (\cos(\theta))^{-1}$).
14. A glucose solution is administered intravenously into the bloodstream at a constant rate r . As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate that is proportional to the concentration at that time. Thus a model for the concentration $C = C(t)$ of the glucose solution in the bloodstream is $\frac{dC}{dt} = r - kC$ where k is a positive constant. Suppose the concentration at time $t = 0$ is C_0. Determine the concentration at any time t by solving the differential equation. If $C_0 < \frac{r}{k}$ what is the concentration after a long time has passed?

15. When a raindrop falls, it increases in size and so its mass at time t is a function of t, $m(t)$. The rate of growth of the mass is $km(t)$ for some positive constant k . When we apply Newton's Law of Motion to the raindrop, we get $(mv)' = gm$, where v is the velocity of the raindrop (directed downward) and g is the acceleration due to gravity. The terminal velocity of the raindrop is $\lim_{t\to\infty} v(t)$. Find an expression of the terminal velocity in terms of g and k.

16. One model for the spread of an epidemic is that the rate of spread is jointly proportional to the number of infected people and the number of uninfected people. In an isolated town of 5000 inhabitants, 160 people have a disease at the beginning of the week and 1200 have it at the end of the week. How long does it take for 80% of the population to become infected?