Moodle Exercise Set 9

Calculus and Linear Algebra II

Spring 2020

1. What is the volume of the parallelopiped spanned by the vectors $v_1 = (3, 2, 1)$, $v_2 = (0, 3, 2)$, $v_3 = (0, 0, 3)$?

2. Let

$$A = \begin{bmatrix} 2 & 5 & 3 & 1 & 1 \\ 1 & 6 & 4 & 0 & 1 \\ 7 & 7 & 3 & 5 & 2 \\ 1 & 1 & 1 & 3 & 0 \\ 8 & 4 & 1 & 0 & 4 \end{bmatrix}.$$

   What is $\det(A)$? *Hint: It's simpler to do this problem using relationship between the rows of the matrix and the determinant. Perhaps pay special attention to the first three rows.*

3. What is the determinant of the $n \times n$ matrix

$$U = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 2 & \cdots & 2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & (n-1) & (n-1) \\ 0 & \cdots & 0 & 0 & n \end{bmatrix}.$$

4. Consider the $n \times n$ matrix C_n with entries which simply count from 1 to n^2. For example $C_3 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. What is the determinant of C_n?

5. Consider the vector space $\mathcal{P}_n$ of polynomials of degree at most n and consider the linear map $D : \mathcal{P}_n \to \mathcal{P}_n$ that maps each polynomial to its derivative. Compute the matrix of D in the basis $\{1, x, x^2, \cdots, x^n\}$ and find its determinant.

6. Let A be an invertible $n \times n$ matrix. If we know that both A and A^{-1} only have integer entries, what are possible values of $\det(A)$? *Hint: Consider how* $\det(A)$ *can be written as a function of entries of* A.

7. Let $u = (2, 3, 5), v = (-1, -4, -10), w = (1, -2, -8)$ be vectors in $\mathbb{R}^3$. Use facts about the determinant to check whether u, v, w are linearly independent.

8. Let $u = (3, 1, 6), v = (2, 0, 4), w = (2, 1, 4)$ be vectors in $\mathbb{R}^3$. Use facts about the determinant to check whether u, v, w are linearly independent.

9. first recall that in general $\det(A + B) \neq \det(A) + \det(B)$. Now let $p, q, r, s \in \mathbb{R}$ and consider the matrices

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -r & -s \\ p & q \end{bmatrix}.$$

   Compute $\det(A) + \det(B)$ and $\det(A + B)$. If these values are equal what is the common value? If not, what is the difference $\det(A + B) - (\det(A) + \det(B))$?

10. Consider the linear transformation given by the matrix.

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}$$

Is T invertible? Why?

11. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that has the following values on the basis elements:

$$T : (1,0,0) \mapsto (-1,0,1)$$
$$T : (0,1,0) \mapsto (3,-2,-1)$$
$$T : (0,0,1) \mapsto (1,1,1)$$

Is T invertible? Why?

12. Let A be a $n \times n$ matrix with entries $a_{i,j}$ consider the matrix B with entries $b_{i,j} = \frac{ia_{i,j}}{j}$. What is $\det(B)$ in terms of $\det(A)$? *Hint: Linearity.*

13. Consider the matrix

$$H = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

What are the cofactors $C_{1,1}$ and $C_{1,2}$? What is $\det(H)$?

14. Consider the matrix

$$H = \begin{bmatrix} 2 & 5 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 0 \\ -1 & 6 & 4 & 0 \end{bmatrix}$$

What are the cofactors $C_{3,4}$ and $C_{4,4}$?

15. Consider the matrix

$$H = \begin{bmatrix} -\lambda & 2 & 7 & 12 \\ 3 & 1-\lambda & 2 & -4 \\ 0 & 1 & -\lambda & 7 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix}$$

where λ is an unknown. Find the $C_{4,4}$ cofactor and compute the determinant of the matrix.