Moodle Exercise Set 11

Calculus and Linear Algebra II

Spring 2020

1. Is the matrix
$$A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$$
diagonalizable?

2. Is the matrix
$$A = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$$
diagonalizable?

3. Let $0 < \mu < 1$ and suppose an $n \times n$ matrix B has characteristic equation
$$\lambda(\lambda - \mu)(\lambda - \mu^2) \cdots (\lambda - \mu^{n-1}).$$
Is B diagonalizable?

4. Consider the vector space $\mathcal{P}_n$ of polynomials of degree at most n and consider the linear map $D : \mathcal{P}_n \to \mathcal{P}_n$ that maps each polynomial to its derivative. First compute the matrix of D in the standard basis $\{1, x, x^2, \dots, x^n\}$ (you might want to refer to problem 5 in online exercise set 9). If $n > 0$, is D diagonalizable?

5. Is the matrix
$$B = \begin{bmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{bmatrix}$$
diagonalizable?

6. Find all values of k that make the matrix
$$A = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & k \\ 0 & 0 & 2 \end{bmatrix}$$
diagonalizable.

7. Consider the 2×2 matrix
$$A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}.$$
Find a formula for A^k.

8. Suppose A is an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Suppose that for all k, $|\lambda_k| < 1$. Compute the limits $\lim_{p\to\infty} A^p$ and $\lim_{p\to\infty} \exp(A^p)$.

9. Recall that when x, y are real numbers, we have the identity $\exp(x) \cdot \exp(y) = \exp(x + y)$. In this exercise we will investigate whether this holds for matrices. Consider the matrices
$$X = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and } Y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$
Compute $\exp(X) \cdot \exp(Y)$ and $\exp(X + Y)$.

10. Consider the matrix

$$A = \begin{bmatrix} 1 & -2-i & 4\exp(i\frac{\pi}{4}) \\ 1+i & i & 2-7i \end{bmatrix}$$

What is $A^\dagger$?

11. A normal matrix U is called *unitary* if $UU^\dagger = I$. Is the matrix

$$U = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

normal and is it unitary?

12. A normal matrix U is called *unitary* if $UU^\dagger = I$. Is the matrix

$$U = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

normal and is it unitary?

13. Find the characteristic polynomial of the matrix

$$H = \begin{bmatrix} -1 & 1-2i & 0 \\ 1+2i & 0 & -i \\ 0 & i & 1 \end{bmatrix}.$$

Plot the polynomial to determine the number of real roots.

14. A matrix H is called *Hermitian* if $H = H^\dagger$. A matrix A is called *anti-Hermitian* if $A = -A^\dagger$. It is possible to write any matrix as a sum of a Hermitian and an anti-Hermitian matrix. Let

$$C = \begin{bmatrix} 1 & -i \\ 2i & 3 \end{bmatrix}$$

and find a Hermitian matrix H and an anti-Hermitian matrix A such that $C = H + A$.

15. **(Bonus problem)** A *graph* is a mathematical structure consisting of a set $V = \{v_1, \ldots, v_n\}$ of vertices and a set E of edges between these vertices. The *adjacency matrix* of a graph is a matrix A such that

$$A_{i,j} = \begin{cases} 1 & \text{there is an edge connecting } i \text{ and } j \\ 0 & \text{otherwise} \end{cases}.$$

A *walk* of length k on a graph is a sequence of vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ such that any two successive vertices have an edge connecting them. More precisely, for any j, v_{i_j} and $v_{i_{j+1}}$ are connected. The total number of walks from vertex v_i to vertex v_j is given by the (i,j) entry of A^k. Consider a graph with three vertices $V = \{v_1, v_2, v_3\}$ such that there are edges between v_1 and v_2, v_2 and v_3, and v_1 and v_3. Find the number of walks of length $k = 10$ starting at v_1 and ending at v_2.