Moodle Exercise Set 12

Calculus and Linear Algebra II

Spring 2020

1. Consider the matrices

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   Check whether $S = LU$ is a valid LU decomposition. If the decomposition is valid, then use L and U to compute $\det(S)$.

2. Compute the LU decomposition of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

   such that all diagonal entries of L are one. What are the diagonal entries of U? What is the entry below the diagonal in L?

3. Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & -3 & 1 \\ 0 & 1 & 2 & 2 \\ -4 & 0 & 9 & 2 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

   Perform LU decomposition on A to obtain lower and upper triangular matrices L and U such that $A = LU$ and L has ones on the diagonal. What are the diagonal elements of U? What are the entries in the diagonal below the main diagonal of L?

4. Consider the system of equations

$$\begin{aligned} 6x + 18y + 3z &= 3 \\ 2x + 12y + z &= 19 \\ 4x + 15y + 3z &= 0. \end{aligned}$$

   Find an LU decomposition of the associated matrix and then solve the system. What is the determinant of the associated matrix? What is the value of xyz?

5. Consider the basis

$$\left\{ v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \right\}$$

   of $\mathbb{R}^3$. Find an orthonormal basis of $\mathbb{R}^3$ containing $v_1/\|v_1\|$ by applying Gram-Schmidt to the basis above.

6. Consider the vector space of polynomials of degree at most 2 spanned by $\{1, x, x^2\}$. On this space, consider the inner product

$$p \cdot q = \int_0^1 p(x)q(x)dx.$$

   Use the Gram-Schmidt procedure on the set of vectors $\{1, x, x^2\}$ to obtain an orthogonal basis that contains the vector 1.

7. Consider the decomposition

$$M = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & -2 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{7}{5} & -\frac{1}{6} \\ 1 & \frac{14}{5} & \frac{1}{3} \\ 0 & 1 & -\frac{7}{6} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{5} & 0 \\ 0 & 1 & -\frac{5}{6} \\ 0 & 0 & 1 \end{bmatrix}.$$

Is this a valid QR decomposition? Why?

8. Find the singular values of the matrix

$$M = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

9. Find the singular values of the matrix

$$M = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}.$$

10. Consider the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$$

and the matrices

$$U = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix}, \quad V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Is the decomposition $A = U\Sigma V^T$ a valid singular value decomposition? If not, why?

11. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Use the Gram-Schmidt procedure to find an orthogonal matrix Q and upper triangular matrix R to obtain the QR-decomposition $A = QR$.