Linear Algebra Review Problems

Calculus and Linear Algebra II

Spring 2020

1. Are the vectors $v_1 = (1, 2, 1)$, $v_2 = (2, 3, -1)$, and $v_3 = (0, 2, 6)$ linearly independent?

2. Let $v = (1, 1, 0)$, $w = (0, -1, 1)$, and $x = (e^2, (e-\pi)(e+\pi), \pi^2)$. Is $x \in \text{span}(v, w)$?

3. Are the vectors $e_1 = (1, 0, 0)$, $e_2 = (1, 1, 0)$, and $e_3 = (1, 1, 1)$ a basis of $\mathbb{R}^3$? If they are not a basis, then what is their span?

4. What is the angle θ between the vectors $v = (0, 2, 2)$ and $w = (\sqrt{2}, 1, 1)$?

5. Consider the vector space $\mathcal{P}_2$ of polynomials of degree at most 2. For two polynomials $p_1, p_2 \in \mathcal{P}_2$ define their scalar product $\langle p_1, p_2 \rangle$to be

$$\langle p_1, p_2 \rangle = \int_0^1 p_1(x) p_2(x)\ dx.$$

   Are $p_1(x) = 2x - 1$ and $p_2(x) = x^2 - x$ orthogonal?

6. Consider the matrix

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

   What transformation does the matrix represent? What is $R_\theta \cdot R_\varphi$?

7. Recall that we can think of an n-dimensional column vector as an $n \times 1$ dimensional matrix. The outer product $v \otimes w$ of two such column vectors v, w is defined as the matrix product

$$v \otimes w = vw^T$$

   where w^T denotes the transpose of w. Compute the outer product

$$v \otimes w = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \otimes \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}.$$

8. Consider the 3×3 matrix M and the vector $v \in \mathbb{R}^3$ given by

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } v = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}.$$

   What is Mv?

9. Consider the 3×3 matrix M and the vector $v \in \mathbb{R}^3$ given by

$$M = \begin{bmatrix} \sqrt{3} & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & \sqrt{3} \end{bmatrix} \text{ and } v = \begin{bmatrix} \sqrt{3} \\ 1 \\ -\sqrt{3} \end{bmatrix}$$

   What is Mv?

10. Consider the linear transformation T on $\mathbb{R}^2$ that first rotates a vector by $\pi/3$ radians in the counter clockwise direction and then doubles its length. What is the matrix of this transformation in the standard basis?

11. Solve the following system of linear equations using Gaussian elimination

$$\begin{aligned} 2x + y - z &= 8, \\ -3x - y + 2z &= -11, \\ -2x + y + 2z &= -3. \end{aligned}$$

12. Solve the following system of linear equations using Gaussian elimination

$$\begin{aligned} 2x - y + z &= 1, \\ 3x + 2y - 4z &= 4, \\ -6x + 3y - 3z &= 2. \end{aligned}$$

13. Compute the rank and nullity of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{bmatrix}.$$

14. Compute the rank and nullity of the matrix

$$A = \begin{bmatrix} -1 & 0 & -1 & 2 \\ 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & -1 \end{bmatrix}$$

15. Find the inverse of the 3×3 matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

16. Find the inverse of the 3×3 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$