Jacobs University
Spring 2020

May 19, 2020

Calculus and Linear Algebra II

Practice Final Exam

Please note that the following instructions apply for the final exam:

- Solve all your exercises on paper, and upload all your notes to LPlus at the end. The grade of the exam will solely be based on your uploaded notes.
- Start a new sheet of paper for each exercise.
- Your notes must be clearly legible.
- Show your work, i.e., carefully write down the steps of your solution. **You will receive points not just based on your final answer, but on the correct steps in your solution.**
- No tools or other resources are allowed for this exam. In particular, no notes and no calculators.
- There are 6 exercises with 15 points each in this exam. 10 points count as bonus points, i.e., your grade will be computed as a percentage based on a total of 80 points.

Note: The practice exam naturally does NOT contain questions about ALL topics of class. Therefore, you may be asked questions about different class topics in the final exam. For example, just because there is no question on LU decompositions in this practice exam, it does not mean there will be no question about that in the final exam.

Problem 1: One-variable Calculus [15 points]

(a) Determine the radius of convergence of the power series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

(b) Let $f(x)$ be infinitely often differentiable at 0. Write down the definition of the (infinite) Taylor series of f around 0.

(c) Compute the Taylor series of $f(x) = \ln(1+x)$ around $x = 0$, for $-1 < x < 1$. (You do NOT need to show that the rest term converges to 0.)

(d) Does the integral $\int_{-2}^{2} x^{-4} dx$ exist as an improper integral (or a sum of improper integrals)? If yes, what is its value? If no, explain why.

Problem 2: Multi-variable Calculus [15 points]

(a) Let

$$\vec{f} : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \mapsto \vec{f}(x, y) = \begin{pmatrix} x^2y \\ \sin(x) \\ e^{x+y} \end{pmatrix}.$$

Determine the Jacobian matrix of $\vec{f}$.

(b) Let $f(x, y) = e^{x^2+y}$. Find the second order Taylor expansion of f around the point $(0, 0)$.

(c) What is the differential $\mathrm{d}f$ of $f(x, y) = \sqrt{x^2 + y^2}$?

Problem 3: Critical Points [15 points]

(a) What are the local minima, local maxima, and saddle points of $f(x, y) = 4 + x^3 + y^3 - 3xy$?

(b) Let $f(x, y) = 1 + xy$. Using the method of Lagrange multipliers, find the critical points of f on the unit circle, i.e., under the constraint $G(x, y) = x^2 + y^2 - 1 = 0$.

Problem 4: Ordinary Differential Equations [15 points]

(a) Consider the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a - by,$$

with parameters $a, b > 0$. Find the solution $y(t)$ for initial condition $y(0) = 0$. What is $\lim_{t\to\infty} y(t)$ for this solution?

(b) Consider the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + 2y + 1.$$

Without solving the ODE explicitly, discuss the qualitative behavior of the solutions, i.e.: What are the equilibrium positions, are they stable or not, how do the solutions behave for large x for different initial conditions?

Problem 5: Determinant, Eigenvalues, and Eigenvectors [15 points]

(a) Compute the determinant of the matrix

$$A = \begin{pmatrix} 2 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 \\ 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

(b) A 4×4 matrix A has determinant $\det(A) = \frac{1}{2}$. Find $\det(2A)$, $\det(-A)$, $\det(A^2)$. Does A have an inverse? If yes, find $\det(A^{-1})$, if no, argue why not.

(c) Compute all eigenvalues and the associated eigenvectors/eigenspace of the matrix

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

and determine what the algebraic and geometric multiplicity of each eigenvalue is.

Problem 6: Special Types of Matrices [15 points]

(a) Let A be an invertible 2×2 matrix with two eigenvalues λ_1 and λ_2. Find the eigenvalues of $B = I + A^{-1}$, where I is the identity matrix.

(b) Recall that the singular values of a real matrix A are the square roots of the eigenvalues of $A^{\mathrm{T}}A$. Compute the singular values of

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 1 \\ -\frac{1}{2} & -\frac{1}{2}. \end{pmatrix}.$$

(c) Determine for each of the following matrices whether they are normal, unitary, orthogonal, Hermitian, real symmetric (several answers can be correct):

(i)

$$\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$

(ii)

$$\begin{pmatrix} 3 & 4+i \\ 4-i & 2 \end{pmatrix},$$

(iii)

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(d) Is the matrix

$$\begin{pmatrix} 6 & -1 & 5 \\ -1 & 2 & 3 \\ 5 & 3 & 0 \end{pmatrix}$$

diagonalizable? Justify your answer. (But you do not need to compute the diagonalization explicitly.)