Jacobs University June 25, 2020
Spring 2020

Calculus and Linear Algebra II

Practice Final Exam 2

Please note that the following instructions apply for the final exam:

- Solve all your exercises on paper, and upload all your notes to LPlus at the end. The grade of the exam will solely be based on your uploaded notes.
- Start a new sheet of paper for each exercise.
- Your notes must be clearly legible.
- Show your work, i.e., carefully write down the steps of your solution. **You will receive points not just based on your final answer, but on the correct steps in your solution.**
- No tools or other resources are allowed for this exam. In particular, no notes and no calculators.
- There are 6 exercises with 15 points each in this exam. 10 points count as bonus points, i.e., your grade will be computed as a percentage based on a total of 80 points.

Note: The practice exam naturally does NOT contain questions about ALL topics of class. Therefore, you may be asked questions about different class topics in the final exam.

Problem 1: One-variable Calculus [15 points]

(a) What is the value of the sum $\sum_{k=0}^{N} x^k$? Show exactly how the resulting formula can be derived. *Hint: Multiply the sum with* $(1-x)$.

(b) For which x does $\sum_{k=0}^{\infty} x^k$ converge? For which x does it diverge?

(c) Let $f(x)$ be infinitely often differentiable at 0. Write down the definition of the (infinite) Taylor series of f around 0.

(d) Let us consider the function $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$, called the hyperbolic sine. Find the Taylor series of $\sinh(x)$ around 0.

Problem 2: Multi-variable Calculus [15 points]

(a) For $f(x, y, z) = xe^y \cos(z)$, compute the the gradient of f as well as $\partial_x \partial_y f$, and $\partial_y \partial_x f$. What is the directional derivative of f at $(0, 0, 0)$ in the direction $\frac{1}{\sqrt{3}}(1, 1, 1)$?

(b) Consider the function $f(x, y) = e^{-x} y^3$, and the point $\vec{a} = (1, 1)$. At $\vec{a}$, in which direction does f increase the most?

(c) Is the differential $df = e^{-x+2y^2}(-dx + 4y dy)$ exact or inexact? In case it is exact, what is f?

Problem 3: Critical Points [15 points]

(a) Find all the critical points of the function $f(x, y) = (1+xy)(x+y)$. (You do not need to state whether they are minima, maxima, or neither.)

(b) Suppose f is two times continuously differentiable, and that $\partial_x\partial_x f(1, 1) = 4$, $\partial_x\partial_y f(1, 1) = 1$, and $\partial_y\partial_y f(1, 1) = 2$. Determine whether f has a local minimum, local maximum, or saddle point at $(1, 1)$.

(c) Use the method of Lagrange multipliers to find the extreme values of the function $f(x, y) = x^2 + y^2$ subject to the constraint $G(x, y) = xy - 1 = 0$.

Problem 4: Ordinary Differential Equations [15 points]

(a) Find the solution $y(x)$ to the first-order ordinary differential equation $\frac{dy}{dx} = \lambda\sqrt{y}$ for any $\lambda > 0$ using separation of variables (it is ok to only find $y(x)$ for $x > 0$). Write down your solution using the general initial data $y(0) = y_0 \geq 0$. How does the solution behave for large x?

(b) Consider the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + 4y + 1.$$

Without solving the ODE explicitly, discuss the qualitative behavior of the solutions, i.e.: What are the equilibrium positions, are they stable or not, how do the solutions behave for large x for different initial conditions?

Problem 5: The Determinant [15 points]

(a) Compute the determinants of the following matrices. Look at these matrices carefully and try to infer what the determinant is with as little computation as possible. Also state for each matrix whether it is invertible or not.

(i)

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 4 & 2 \\ 1 & 5 & 2 \end{pmatrix},$$

(ii)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 525 & 725 \\ 0 & 0 & 1 & 352 \\ 0 & 0 & 0 & 21 \end{pmatrix},$$

(iii)

$$\begin{pmatrix} 21 & 43 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 72 & 2 \\ -1 & 0 \end{pmatrix}$$

(Here it is asked for the determinant of the product of these two matrices, and whether or not this product has an inverse).

(b) Consider the linear system of equations

$$\begin{aligned} 3x_1 + 2x_2 + 5x_3 &= 8, \\ x_1 + 2x_2 + 2x_3 &= 5, \\ 2x_1 + 2x_2 + 3x_3 &= 7. \end{aligned}$$

Use Cramer's rule to determine the solution x_3.

(c) Consider the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Compute all eigenvalues, all eigenvectors, and the algebraic and geometric multiplicities of the eigenvalues. Is this matrix diagonalizable?

Problem 6: Eigenvalues, Eigenvectors, and Matrix Decompositions [15 points]

(a) Let

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

State whether or not A is orthogonal, unitary, real symmetric, Hermitian, normal. (A can be several of those.) Determine all eigenvectors and eigenvalues of A. Is A diagonalizable? If yes, write down a diagonalization $A = V\Lambda V^{-1}$, where Λ is a diagonal matrix, and V is invertible. Can we choose V unitary?

(b) Compute an LU decomposition of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Choose L such that it has only 1's on the diagonal.