Jacobs University
Spring 2020

May 4, 2020

Foundations of Mathematical Physics

Homework 10

Due on May 11, 2020

Problem 1 [6 points]: Laplace operator on the half line

(a) We consider the operator $(-\Delta, C_0^\infty((0,\infty)))$ with $\Delta = \frac{d^2}{dx^2}$ as an unbounded operator on the Hilbert space $L^2([0,\infty))$. Is this operator self-adjoint? Is it essentially self-adjoint? Do self-adjoint extensions exist? Prove your answers using theorems from class. What is the adjoint operator?

(b) We define

$$H_D^2(0,\infty) := \{\psi \in H^2(0,\infty) : \psi(0) = 0\},$$
$$H_N^2(0,\infty) := \{\psi \in H^2(0,\infty) : \psi'(0) = 0\},$$

and call $-\Delta_D = (-\Delta, H_D^2(0,\infty))$ the Dirichlet Laplacian and $-\Delta_N = (-\Delta, H_N^2(0,\infty))$ the Neumann Laplacian. Show that these operators are self-adjoint. Are they self-adjoint extensions of the operator from (a)? How do the unitary groups they generate differ? (Consider the reflection of a wave packet at the origin. How does one get solutions of the Schrödinger equation with the specified boundary conditions from solution on $\mathbb{R}$?)

(c) Try to specify all self-adjoint extensions of the operator from (a) (using intuition from physics, if necessary).

Problem 2 [6 points]: δ-interaction in $\mathbb{R}^3$

On the Hilbert space $L^2(\mathbb{R}^3)$, we consider the operator $H_0 = -\Delta$ with domain $D(H_0) = C_0^\infty(\mathbb{R}^3 \setminus \{0\})$.

(a) Do the equations $-\Delta\varphi_\pm = \pm i\varphi_\pm$ have solutions in the sense of distributions?

(b) We now consider the equation from (a) as an ordinary differential equation and look for spherically symmetric solutions, i.e., for solutions $\varphi_\pm(r)$ of the equations

$$-\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\varphi_\pm(r) = \pm i\varphi_\pm(r).$$

Find all solutions to these equations. Do they have solutions in $L^2(\mathbb{R}^3)$?

(c) Are the solutions from (b) also solutions to $H_0^* \varphi_\pm = \pm i\varphi_\pm$? (*Hint: Check whether* $\langle \varphi_\pm, H_0\psi \rangle = \langle \pm i\varphi_\pm, \psi \rangle$ *for all* $\psi \in D(H_0)$.)

(d) Find the deficiency spaces and deficiency indices. Does $(H_0, D(H_0))$ have self-adjoint extensions?

(e) Is $(H_0, H^2(\mathbb{R}^3))$ a self-adjoint extension? The Cayley transform would allow us to find all self-adjoint extensions, but since we have not studied it carefully, we omit this exercise here. Can you find other self-adjoint extensions anyway (for extra points)?

Problem 3 [4 points]: Von Neumann theorem

Let $(H, D(H))$ be a densly defined symmetric linear operator. Prove that if there is a conjugation C with $CD(H) \subset D(H)$ and $CH\varphi = HC\varphi$ for all $\varphi \in D(H)$, then $N^+ = N^-$, i.e., the deficiency indices of H coincide.

Problem 4 [4 points]: Operator bounds

Let A, B be densly defined linear operators with $D(A) \subset D(B)$. Prove that the following two statements are equivalent:

(i) There are $a, b \geq 0$ such that

$$\|B\varphi\| \leq a\,\|A\varphi\| + b\,\|\varphi\|$$

for all $\varphi \in D(A)$.

(ii) There are $\tilde{a}, \tilde{b} \geq 0$ such that

$$\|B\varphi\|^2 \leq \tilde{a}^2\,\|A\varphi\|^2 + \tilde{b}\,\|\varphi\|^2$$

for all $\varphi \in D(A)$.

Prove also that the infimum over all permissible a in (i) coincides with the infimum over all permissible $\tilde{a}$ in (ii).