Jacobs University
Spring 2020

March 9, 2020

Foundations of Mathematical Physics

Homework 4

Due on March 16, 2020

Problem 1 [7 points]: Heat Equation

(a) Let $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$. Determine the solution to the heat equation

$$\begin{aligned} \partial_t \psi(t,x) &= \Delta_x \psi(t,x) \qquad \text{for all } (t,x) \in [0,\infty) \times \mathbb{R}^d, \\ \psi(0,x) &= \psi_0(x) \qquad \text{for all } x \in \mathbb{R}^d \end{aligned}$$

by using the Fourier transform. Write the solution as

$$\psi(t,x) = \int_{\mathbb{R}^d} K(t, x-y)\psi_0(y)dy, \tag{1}$$

and explicitly state what the function $K : (0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ is.

(b) Let $\psi_0 \in C(\mathbb{R}^d)$ be bounded. Show that Equation (1) defines a bounded function $\psi \in C^\infty((0,\infty) \times \mathbb{R}^d)$ which solves the heat equation on $(0,\infty) \times \mathbb{R}^d$. Show also that ψ can be continuously extended by ψ_0 at $t = 0$, i.e., show that $\lim_{t\to 0} \psi(t,x) = \psi_0(x)$ for all $x \in \mathbb{R}^d$. (*Hint: Use Problem 4 from Homework 2.*)

Problem 2 [6 points]: Away from the support of $\widehat{\psi}$

Let $\psi_0 \in \mathcal{S}$ and let its Fourier transform have compact support, i.e., $K = \operatorname{supp}(\widehat{\psi})$ is compact. Let U be an open ε-neighborhood of K, i.e., the distance between the complement of U and K is ε, i.e., $\operatorname{dist}(U^c, K) = \varepsilon > 0$. Prove that then for any $m \in \mathbb{N}$ there is a constant C_m such that for any t, x with $x/t \notin U$ and $|t| \geq 1$,

$$\left| \left(e^{-it(-\Delta)/2} \psi_0 \right)(t,x) \right| \leq C_m \left(1 + |t|\right)^{-m}.$$

Hint: One could write the phase factor as $e^{i\alpha S}$ *with* $S(k) = \frac{kx - k^2 t/2}{1+|t|}$, *prove that*

$$e^{i\alpha S} = \left[\frac{1}{i\alpha} |\nabla_k S|^{-2} (\nabla_k S) \nabla_k \right]^m e^{i\alpha S},$$

and then integrate by parts.

Problem 3 [3 points]: Uncertainty on $\mathcal{S}$
Prove the Heisenberg uncertainty principle. Let

$$(\delta x_j)^2 := \langle \psi, \big(x_j - \langle \psi, x_j \psi\rangle\big)^2 \psi\rangle \qquad (\delta p_j)^2 := \langle \psi, \big(p_j - \langle \psi, p_j \psi\rangle\big)^2 \psi\rangle$$

be the variances in the position and the asymptotic momentum distributions, where $p_j = -i\partial_{x_j}$ and $\langle f, g\rangle = \int \overline{f} g$. (Let's just take $\psi \in \mathcal{S}$ here.) Prove that

$$\delta x_j \delta p_j \geq \frac{\|\psi\|^2}{2}.$$

Problem 4 [4 points]: Refined Uncertainty on $\mathcal{S}$
Prove the refined uncertainty principle (Hardy's inequality) on $\mathcal{S}(\mathbb{R}^3)$, i.e., that

$$\langle \psi, (-\Delta)\psi\rangle \geq \frac{1}{4}\langle \psi, |x|^{-2}\psi\rangle$$

for all $\psi \in \mathcal{S}(\mathbb{R}^3)$. *Hint: Look at the quantity* $\big[|x|^{-1} p_j |x|^{-1}, x_j\big]$, *where* $[A, B] := AB - BA$ *is the commutator. Note: This could be directly used to give a proof of the stability of hydrogen atoms.*