Jacobs University — March 16, 2020
Spring 2020

Foundations of Mathematical Physics

Homework 5

Due on March 23, 2020

Problem 1 [5 points]: Orthonormal basis
Prove that an orthonormal sequence $(\varphi_j)_j$ in a Hilbert space is an orthonormal basis if and only if

$$\langle \varphi_j, \psi \rangle = 0 \quad \text{for all } j \in \mathbb{N} \quad \Rightarrow \quad \psi = 0.$$

Problem 2 [7 points]: Operator norm
Prove the following lemma that was stated in class: Let $\mathcal{L}(X, Y)$ be the set of bounded linear operators from $X \to Y$. Then $\mathcal{L}(X, Y)$ with the norm

$$\|L\|_{\mathcal{L}(X,Y)} := \sup_{\|x\|_X = 1} \|Lx\|_Y$$

is a normed space. Furthermore, if Y is a Banach space, then so is $\mathcal{L}(X, Y)$.

Problem 3 [8 points]: Riemann-Lebesgue lemma

(a) Prove that

$$C_\infty(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) : \lim_{R\to\infty} \sup_{|x|>R} |f(x)| = 0\}$$

with the supremum norm is a Banach space. You can use that the space of continuous bounded functions $C_b(\mathbb{R}^d)$ with the supremum norm is a Banach space.

(b) Now prove the Riemann-Lebesgue lemma, i.e., prove that $\mathcal{F}L^1(\mathbb{R}^d) \subset C_\infty(\mathbb{R}^d)$. In order to do so you could first prove continuity of $\mathcal{F} : (\mathcal{S}(\mathbb{R}^d), \|\cdot\|_{L^1(\mathbb{R}^d)}) \to (C_\infty(\mathbb{R}^d), \|\cdot\|_\infty)$. Then you could use that $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d)$ and Theorem 3.20 from class in order to continuously extend $\mathcal{F}$ in $L^1(\mathbb{R}^d)$. Why does this extension agree with the usual formulas for $\mathcal{F}$ on all of $L^1(\mathbb{R}^d)$?