Foundations of Mathematical Physics

Session 1
Feb. 3, 2020

Organization:

- class: Mon/Wed, 11:15-12:30 (WH 8)
- website: lecture notes, homework, news
- grade: 100% final exam
- TA: ?
- books: • lecture notes by Stefan Teufel (U Tübingen), German only
  • see references on website

Topics:

Generally: Mathematics of (non-relativistic) quantum mechanics (QM)

Fields we touch upon: • Analysis
• Functional Analysis
• PDEs
• maybe later: • Harmonic Analysis
• Many-body QM (• Lie groups)

Structure:

- short introduction to quantum mechanics
- Fourier transform, distributions, free Schrödinger equation
- Hilbert space, self-adjoint operators, unitary groups, interacting Schrödinger eq.
- spectral theorem
- maybe: non-linear Schrödinger eq., second quantization, Bose-Einstein condensates

1. Introduction

1.1 Motivation

classical physics: • Newton's eq.: $F = m \cdot a$

↳ initial position and momentum determine trajectories of point particles

• with special (general) relativity: locality

↳ "nothing moves faster than the speed of light c"

problems at beginning of 20th century: • blackbody radiation

• photoelectric effect

• stability of atoms / matter

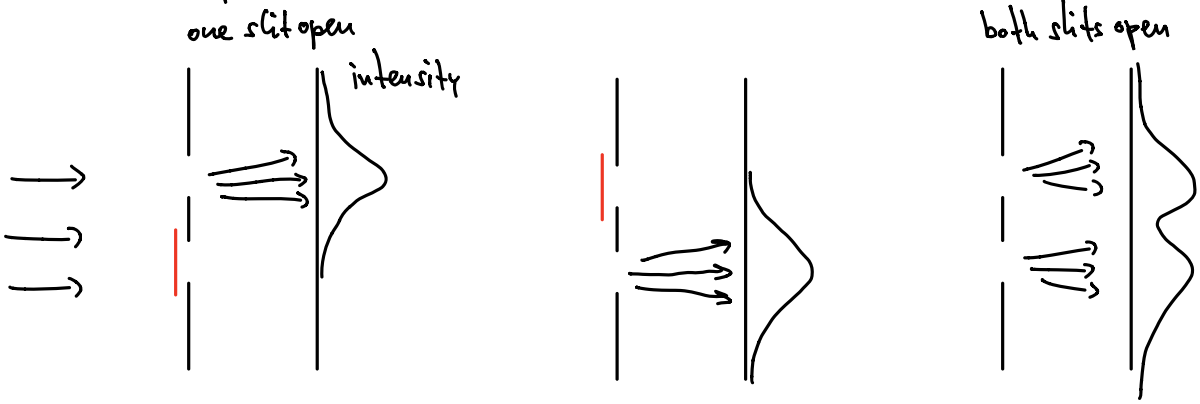
solution to these problems led to Quantum Mechanics (QM)

↳ physics: Planck, Einstein, Schrödinger, Bohr, Heisenberg, Dirac, Born, de Broglie

↳ math: Hilbert, von Neumann

Example of "weird" quantum behavior: double slit experiment

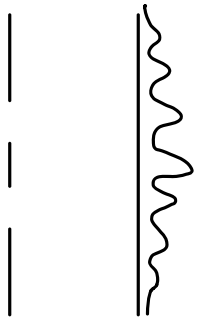
classical particles:



electromagnetic waves (light), water waves

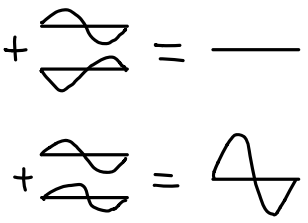
single slit similar

both slits open



interference pattern

explanation:

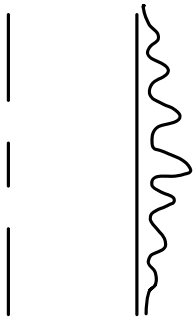


destructive interference

constructive interference

electrons:

both slits open

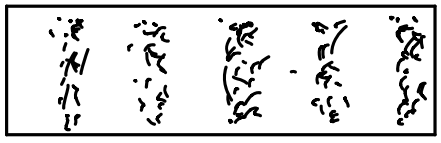


interference, big surprise!

maybe a collective wave?

single separated electrons: (both slits open)

side view



single localized detection events => electrons are particles

interference pattern => electrons are waves and go through both slits at the same time

=> seems like a paradox

Resolution of this paradox still hotly debated

Not debated: probabilistic description: some wave describes probability distribution of particle positions

1.2 Single Particle QM

d = dimension of space ($d = 1, 2, 3$ usually)

Wave function $\psi : \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{C}, \ (t, x) \mapsto \psi(t, x)$

normalization: $\int_{\mathbb{R}^d} |\psi(t, x)|^2 \, dx =: \|\psi(t, \cdot)\|^2_{L^2(\mathbb{R}^d)} = 1$

$|\psi(t, x)|^2 =: \rho(t, x)$ = probability density for particle to be at position x at time t

$\Lambda \subset \mathbb{R}^d \Rightarrow \mathbb{P}(Q(t) \in \Lambda) \equiv \mathbb{P}^{\psi_t}(\Lambda) = \int_\Lambda |\psi(t, x)|^2 \, dx$

↳ probability that particle is in Λ (at time t)

note: ρ is only a probability density, not a charge or mass density!