Session 5
Feb. 17, 2020

recall: • a $\boxed{\text{multi-index}}$ $\alpha \in \mathbb{N}_0^d$ is a tupel $(\alpha_1, \dots, \alpha_d)$, $\alpha_j \in \mathbb{N}_0$.

We denote $|\alpha| := \sum_{j=1}^d \alpha_j$, and for $x \in \mathbb{R}^d$, $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, $\partial_x^\alpha := \dfrac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}}$.

• $\boxed{C^p(\mathbb{R}^d)} := \Big\{ f : \underbrace{\partial_x^\alpha f \text{ continuous } \forall \text{ multi-indices } \alpha \text{ with } |\alpha| \leq p}_{f \text{ p times continuously differentiable}} \Big\}$

• $\boxed{C^\infty(\mathbb{R}^d)} = \bigcap_{p \in \mathbb{N}} C^p(\mathbb{R}^d)$ = smooth functions (∞ often continuously differentiable)

• $C^0(\mathbb{R}^d) = \boxed{C(\mathbb{R}^d)}$ = continuous functions

• $\underbrace{\boxed{C_\infty(\mathbb{R}^d)}}_{\text{sometimes called } C_0(\mathbb{R}^d)} := \Big\{ f \in C(\mathbb{R}^d) : \underbrace{\lim_{|x| \to \infty} f(x) = 0}_{\text{more exact: } \lim_{R \to \infty} \sup_{|x| > R} |f(x)| = 0} \Big\}$

• $\boxed{C_c^p(\mathbb{R}^d)} := C^p(\mathbb{R}^d) \cap \{ f : \operatorname{supp} f \text{ compact} \}$ = functions with compact support

Lemma 2.3: Riemann-Lebesgue

$$f \in L^1(\mathbb{R}^d) \Rightarrow \hat{f} \in C_\infty(\mathbb{R}^d)$$

Proof: • $f \in L^1(\mathbb{R}^d)$, recall $\hat{f}(k) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \underbrace{f(x) e^{-ikx}} dx$

↳ cont. in k for a.a. x

↳ $\sup_k |f(x) e^{-ikx}| = |f(x)| \in L^1(\mathbb{R}^d)$

$\Rightarrow \hat{f}$ continuous with Lemma 2.2.

• decay at ∞ relatively easy to show, follows later from a more general result

$\square$

next: a class of "very nice" functions

Definition 2.5: Schwartz space

We call the $\mathbb{C}$-vector space

$$S(\mathbb{R}^d) := \{ f \in C^\infty(\mathbb{R}^d) : \|f\|_{\alpha,\beta} < \infty \ \forall \text{ multi-indices } \alpha, \beta \in \mathbb{N}_0^d \}$$ Schwartz space

(space of smooth rapidly decaying functions), where

$$\|f\|_{\alpha,\beta} := \|x^\alpha \partial_x^\beta f(x)\|_\infty = \sup_{x \in \mathbb{R}^d} |x^\alpha \partial_x^\beta f(x)| .$$

note: • for $f \in S(\mathbb{R}^d)$, f and all partial derivatives decay faster than any polynomial

• e.g., $e^{-x^2} \in S(\mathbb{R}^d)$, $C_c^\infty(\mathbb{R}^d) \subset S(\mathbb{R}^d)$

Definition: On a vector space V, a map $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ is called semi-norm if

- $\|\lambda f\| = |\lambda| \cdot \|f\|$
- $\|f+g\| \leq \|f\| + \|g\|$

note: • for a norm, we require additionally that $\|f\| = 0 \Rightarrow f = 0$

• $\|f\|_{\alpha,\beta}$ are semi-norms (for $\beta = 0$, $\|f\|_{\alpha,0}$ is also a norm)

next: since we have only semi-norms on S, it is not a Banach space; but we can construct a complete metric space (Fréchet space) in the following way.

Lemma 2.8:

$$d_S(f,g) := \sum_{n=0}^{\infty} 2^{-n} \sup_{|\alpha|+|\beta|=n} \left(\frac{\|f-g\|_{\alpha,\beta}}{1+\|f-g\|_{\alpha,\beta}} \right)$$ is a metric on S.

Proof: first, note that $\frac{x}{1+x}$ maps $\mathbb{R}_{\geq 0}$ to $[0,1]$ and is monotonically increasing.

$$\Rightarrow d_S(f,g) \leq \sum_{n=0}^{\infty} 2^{-n} = \frac{1}{1-\frac{1}{2}} = 2$$

we check: • $d_S(f,g) \geq 0$ clear

• $d_S(f,g) = d_S(g,f)$ clear

• $d_S(f,g) = 0 \Leftrightarrow f=g$?

↳ "$\Leftarrow$" clear

↳ "$\Rightarrow$" $d_S(f,g)=0$

$\Rightarrow$ in particular $\|f-g\|_{0,0} = \|f-g\|_{\infty} := \sup_{x\in\mathbb{R}^d} |f(x)-g(x)| = 0 \quad \Rightarrow f=g$

• $d_S(f,g) \leq d_S(f,h) + d_S(h,g)$?

triangle inequality holds for $\|\cdot\|_{\alpha,\beta}$ and

$$\frac{x+y}{1+x+y} = \frac{x}{1+x+y} + \frac{y}{1+x+y} \leq \frac{x}{1+x} + \frac{y}{1+y}$$ $\square$

Corollary: convergence in S

$$f_n \xrightarrow{n\to\infty} f \text{ in } S \Leftrightarrow d_S(f,f_n) \xrightarrow{n\to\infty} 0$$

$$\Leftrightarrow \|f-f_n\|_{\alpha,\beta} \xrightarrow{n\to\infty} 0 \quad \forall \alpha,\beta \in \mathbb{N}_0^d.$$

Lemma 2.9: The metric space (S, d_S) is complete.

Proof: Let $(f_m)_m$ be a Cauchy sequence in S $(\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \text{ s.t. } d(f_m, f_n) < \varepsilon \; \forall m, n > N)$

$\Rightarrow$ (f_m) is also Cauchy w.r.t. all $\|\cdot\|_{\alpha,\beta}$

$\Rightarrow$ $f_m^{(\alpha,\beta)}(x) := x^\alpha \partial_x^\beta f_m(x)$ is Cauchy w.r.t. $\|\cdot\|_\infty$

since $C_b(\mathbb{R}^d) := \{ f \in C(\mathbb{R}^d) : f \text{ bounded} \}$ is complete w.r.t. $\|\cdot\|_\infty$ (Analysis II),

$f_m^{(\alpha,\beta)} \xrightarrow{m \to \infty} f^{(\alpha,\beta)}$ uniformly

Next, we def. $f := f^{(0,0)}$. To show: $f \in C^\infty(\mathbb{R}^d)$ and $x^\alpha \partial_x^\beta f = f^{(\alpha,\beta)}$

(since then $f \in S(\mathbb{R}^d)$ and $d_S(f_m, f) \xrightarrow{m \to \infty} 0$).

Here, let us just show for $d=1$ that $f \in C^1(\mathbb{R}^d)$ and $\partial_x f = f^{(0,1)}$, the rest goes analogously.

$f_m \in S(\mathbb{R}) \Rightarrow \quad f_m(x) = f_m(0) + \int_0^x f_m'(y)\,dy$

since $f_m \to f$ and $f_m' \to f^{(0,1)}$ <u>uniformly</u>, we get

$$\lim_{m \to \infty} f_m(x) = f(x) = f(0) + \underbrace{\lim_{m \to \infty} \int_0^x f_m'(y)\,dy}$$

$$= \int_0^x f^{(0,1)}(y)\,dy \text{ due to uniform convergence}$$

$\Rightarrow f \in C^1(\mathbb{R})$ and $f' = f^{(0,1)}$ $\square$