Session 6
Feb. 19, 2020

Last time: • $\mathcal{S}(\mathbb{R}^d) = \{ f \in C^\infty(\mathbb{R}^d) : \underbrace{\|f\|_{\alpha,\beta}}_{\text{family of semi-norms}} := \|x^\alpha \partial_x^\beta f(x)\|_\infty < \infty \ \forall \alpha,\beta \in \mathbb{N}_0^d \}$

• $(f_n)_n$ converges in $\mathcal{S}$: $\|f_n - f\|_{\alpha,\beta} \xrightarrow{n\to\infty} 0 \ \forall \alpha,\beta \in \mathbb{N}_0^d$

• $(\mathcal{S}, d_{\mathcal{S}})$ is a complete metric space

Lemma 2.11: Properties of the Fourier transform

(1) $\mathcal{F}$ and $\mathcal{F}^{-1}$ are continuous linear maps $\mathcal{S} \to \mathcal{S}$

(2) $\forall \alpha,\beta \in \mathbb{N}_0^d$, $f \in \mathcal{S}$: $\left((ik)^\alpha \partial_k^\beta \mathcal{F} f\right)(k) = \left(\mathcal{F} \partial_x^\alpha (-ix)^\beta f\right)(k)$,

in particular: $\widehat{(xf)}(k) = i(\nabla_k \hat{f})(k)$ and $\widehat{(\nabla_x f)}(k) = ik\hat{f}(k)$

Proof:

(2) recall $(\mathcal{F}f)(k) = \hat{f}(k) = (2\pi)^{-\frac{d}{2}} \int f(x) e^{-ikx} dx$

with Lemma 2.2: $\left((ik)^\alpha \partial_k^\beta \mathcal{F}f\right)(k) = (2\pi)^{-\frac{d}{2}} \int (ik)^\alpha \partial_k^\beta e^{-ikx} f(x) dx$

$$= (2\pi)^{-\frac{d}{2}} \int (ik)^\alpha (-ix)^\beta e^{-ikx} f(x) dx$$

note: all integrals exist since $f \in \mathcal{S}$

$$= (2\pi)^{-\frac{d}{2}} (-1)^{|\alpha|} \int \left(\partial_x^\alpha e^{-ikx}\right) (-ix)^\beta f(x) dx$$

$|\alpha|$-times integration by parts (boundary terms vanish, since $f \in \mathcal{S}$)

$$= (2\pi)^{-\frac{d}{2}} \int e^{-ikx} \left(\partial_x^\alpha (-ix)^\beta f(x)\right) dx$$

$$= \mathcal{F}\left(\partial_x^\alpha (-ix)^\beta f\right)(k)$$

(1) $$\|\hat{f}\|_{\alpha,\beta} = \|k^\alpha \partial_x^\beta \hat{f}\|_\infty \overset{\text{(2) and triangle inequality}}{\leq} (2\pi)^{-\frac{d}{2}} \int |\partial_x^\alpha x^\beta f(x)|\, dx$$

$$= (2\pi)^{-\frac{d}{2}} \int (1+|x|^2)^d \,|\partial_x^\alpha x^\beta f(x)|\, (1+|x|^2)^{-d}\, dx$$

$$\leq (2\pi)^{-\frac{d}{2}} \left(\sup_{x\in\mathbb{R}^d} (1+|x|^2)^d |\partial_x^\alpha x^\beta f(x)|\right) \underbrace{\int (1+|x|^2)^{-d}\, dx}_{\substack{= \text{const.} \cdot \int_0^\infty (1+r^2)^{-d} r^{d-1}\, dr \\ \leq C \quad (\text{since integrand} \sim r^{-2d+d-1} \text{ for large } r)}}$$

$$\leq \tilde{C} \sum_{j=0}^{m} \sup_{|\tilde{\alpha}|+|\tilde{\beta}|=j} \|f\|_{\tilde{\alpha},\tilde{\beta}} \quad \text{for some } m\in\mathbb{N},\ \tilde{C}>0$$

$\Rightarrow \mathcal{F}f \in \mathcal{S}$ and $f_n \to f$ in $\mathcal{S}$ implies $\mathcal{F}f_n \to \mathcal{F}f$ in $\mathcal{S}$,

meaning $\mathcal{F}: \mathcal{S} \to \mathcal{S}$ is continuous (on metric spaces: continuity $\Leftrightarrow$ def. of cont. in terms of sequences)

similar for $\mathcal{F}^{-1}$ $\square$

Theorem 2.12:

$\mathcal{F}: \mathcal{S} \to \mathcal{S}$ is a continuous bijection with continuous inverse $\mathcal{F}^{-1}$.

Proof: HW: (1) show $\mathcal{F}^{-1}\mathcal{F} = \text{id}$ only on C_c^∞ = smooth fct.s with compact support

↳ consider $\operatorname{supp} f \subset [-m,m]^d$

$\Rightarrow$ Fourier series, write f as Riemann sum

(2) show that C_c^∞ is dense in $\mathcal{S}$, then thm. follows from continuity

(use some smooth cutoff function)

Lemma 2.14: Plancherel on $\mathcal{S}$

For $f, g \in \mathcal{S}$, we have $\int \hat{f}(x) g(x)\, dx = \int f(x)\, \hat{g}(x)\, dx$, and, in particular,

$$\int |\hat{f}(k)|^2\, dk = \int |f(x)|^2\, dx.$$

Proof: simple computation, HW.

back to $i\partial_t \psi(t,x) = -\frac{1}{2}\Delta_x \psi(t,x)$

formally we solve this by applying $\mathcal{F}$: $i\partial_t \hat{\psi}(t,k) = -\frac{1}{2}(\mathcal{F}\Delta_x \psi)(t,k) = \frac{1}{2}k^2 \hat{\psi}(t,k)$

$\Rightarrow \hat{\psi}(t,k) = e^{-i\frac{k^2}{2}t}\hat{\psi}(0,k)$ unique global solution

$\Rightarrow \psi(t,x) = \left(\mathcal{F}^{-1} e^{-i\frac{k^2}{2}t}\mathcal{F}\psi_0\right)(x)$, $\psi_0(x) = \psi(0,x)$ is the initial condition

Theorem 2.16: Solution to free SE in $\mathcal{S}$

Let $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$. Then the unique global solution $\psi \in C^\infty(\mathbb{R}_t, \mathcal{S}(\mathbb{R}^d))$ to the free SE with $\psi(0,x) = \psi_0(x)$ is, for $t \neq 0$,

$$\psi(t,x) = (2\pi i t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{2t}} \psi_0(y)\, dy.$$

Furthermore, $\|\psi(t,\cdot)\|_{L^2(\mathbb{R}^d)} = \|\psi_0\|_{L^2(\mathbb{R}^d)}$ $\forall t \in \mathbb{R}$

Important note: What does $\boxed{\psi \in C^\infty(\mathbb{R}_t, \mathcal{S}(\mathbb{R}^d))}$ mean?

It means first that $\psi(t,\cdot) \in \mathcal{S}$, i.e., $\psi(t,x)$ as a function of x is in $\mathcal{S}$, but also the map $\psi: \mathbb{R}_t \to \mathcal{S}(\mathbb{R}^d)$ is ∞-often differentiable.

Proof:

The formula $\psi(t,x) = \left(\mathcal{F}^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F}\psi_0\right)(x) = (2\pi i t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{2t}} \psi_0(y)\,dy$

can be checked by direct computation (use Fourier transform of Gaussian).

Next: Let us show that $t \mapsto \psi(t,\cdot)$ is differentiable once, then $\psi \in C^\infty(\mathbb{R}, \mathcal{S})$ follows by repeating the argument.

guess: derivative is $\dot\psi(t,x) = -i\left(\mathcal{F}^{-1} \frac{k^2}{2} e^{-i\frac{k^2}{2}t} \mathcal{F}\psi_0\right)(x)$, which we know is in $\mathcal{S}(\mathbb{R}^d)$.

To show: $\lim_{h\to 0} \left\| \frac{\psi(t+h)-\psi(t)}{h} - \dot\psi(t) \right\|_{\alpha,\beta} = 0 \quad \forall \alpha,\beta \in \mathbb{N}_0^d$

by continuity of $\mathcal{F}$ (Lemma 2.11), this is equivalent to

$$\underbrace{\lim_{h\to 0} \left\| \mathcal{F}\left(\frac{\psi(t,t+h)-\psi(t)}{h} - \dot\psi(t) \right) \right\|_{\alpha,\beta}} = 0 \quad \forall \alpha,\beta \in \mathbb{N}_0^d$$

$$= \lim_{h\to 0} \left\| \frac{\hat\psi(t+h)-\hat\psi(t)}{h} - \hat{\dot\psi}(t) \right\|_{\alpha,\beta}$$

$$= \lim_{h\to 0} \sup_{k\in\mathbb{R}^d} \left| k^\alpha \partial_k^\beta \left(\frac{e^{-i\frac{k^2}{2}(t+h)} - e^{-i\frac{k^2}{2}t}}{h} + i\frac{k^2}{2} e^{-i\frac{k^2}{2}t} \right) (\mathcal{F}\psi_0)(k) \right| = 0,$$

since $\mathcal{F}\psi_0 \in \mathcal{S}$ and $e^{-i\frac{k^2}{2}t}$ smooth. Proof of $\|\psi(t,\cdot)\|_{L^2} = \|\psi_0\|_{L^2}$ next time. $\square$