Session 7
Feb. 24, 2020

recall: solution $\overbrace{\Psi \in C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}^d))}^{\Psi : \mathbb{R} \to \mathcal{S},\ t \mapsto \Psi(t) \text{ is } \infty\text{-often differentiable}}$ of $i\partial_t \Psi(t,x) = -\frac{1}{2}\Delta_x \Psi(t,x)$

is $\Psi(t,x) = \left(\mathcal{F}^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F}\Psi_0\right)(x) = (2\pi i t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{2t}} \Psi_0(y)\,dy.$

also: $\|\Psi(t,\cdot)\|_{L^2(\mathbb{R}^d)} = \|\Psi_0\|_{L^2(\mathbb{R}^d)} \quad \forall t \in \mathbb{R}$

$\hookrightarrow$ proof: $\|\Psi(t,\cdot)\|_{L^2}^2 = \int |\Psi(t,x)|^2\,dx = \int \left|\left(\mathcal{F}^{-1} e^{-i\frac{k^2}{2}t}\mathcal{F}\Psi_0\right)(x)\right|^2 dx$

$$\overset{\text{Plancherel (2.14)}}{=} \int \left|e^{-i\frac{k^2}{2}t}\mathcal{F}\Psi_0(x)\right|^2 dx$$

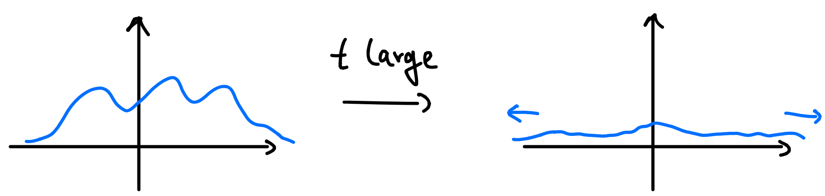
$$= \int |\mathcal{F}\Psi_0(x)|^2\,dx$$

$$\overset{\text{Plancherel (2.14)}}{=} \int |\Psi_0(x)|^2\,dx = \|\Psi(0,\cdot)\|_{L^2}^2$$

Note: $\|\Psi(t,\cdot)\|_\infty = \sup_{x\in\mathbb{R}^d} |\Psi(t,x)| = \sup_{x\in\mathbb{R}^d} \left|(2\pi i t)^{-\frac{d}{2}} \int e^{i\frac{|x-y|^2}{2t}} \Psi_0(y)\,dy\right|$

$$\le (2\pi t)^{-\frac{d}{2}} \|\Psi_0\|_{L^1} \xrightarrow{t\to\infty} 0$$

$\Rightarrow$ wave functions spread:



next: want to consider multiplication operators $\psi(x) \mapsto f(x)\psi(x)$

Definition 2.18: Smooth polynomially bounded functions

$$C^\infty_{pol}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) : \forall \alpha \in \mathbb{N}_0^d \; \exists n_\alpha \in \mathbb{N} \text{ and } C_\alpha < \infty \text{ s.t. } |\partial_x^\alpha f(x)| \le C_\alpha (1+|x|^2)^{\frac{n_\alpha}{2}} \right\}$$

Note: • notation: $(1+|x|^2)^{\frac{1}{2}} =: \langle x \rangle$

• e.g., all polynomials $\in C^\infty_{pol}$, $e^{ikx} \in C^\infty_{pol}$, $e^x \notin C^\infty_{pol}$

Lemma: For $f \in C^\infty_{pol}(\mathbb{R}^d)$, the multiplication operator $M_f : \mathcal{S} \to \mathcal{S}$, $\psi(x) \mapsto f(x)\psi(x)$ is continuous.

Proof: clear: if $\|\psi_n - \psi\|_{\alpha,\beta} \xrightarrow{n\to\infty} 0 \;\forall \alpha,\beta$, then also

$$\|M_f(\psi_n - \psi)\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^d} |x^\alpha \partial_x^\beta (f(x)(\psi_n(x) - \psi(x)))| \xrightarrow{n\to\infty} 0 \qquad \square$$

Definition 2.19:

For $f \in C^\infty_{pol}(\mathbb{R}^d)$ we define the pseudo-differential operator

$$f(-i\nabla) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d),\; \psi(x) \longmapsto \left(f(-i\nabla_x)\psi\right)(x) = \left(\mathcal{F}^{-1} M_f \mathcal{F}\psi\right)(x) = \left(\mathcal{F}^{-1} f(k) \mathcal{F}\psi\right)(x)$$

Note: • $f(-i\nabla)$ continuous, since $M_f, \mathcal{F}, \mathcal{F}^{-1}$ continuous

• $f(k) = k^\alpha \Rightarrow f(-i\nabla) = (-i)^{|\alpha|} \partial_x^\alpha$ is the usual differential operator

Examples:

- semi-relativistic or pseudo-relativistic Schrödinger equation: $i\partial_t \psi(t,x) = \sqrt{1-\Delta}\, \psi(t,x)$

  ↳ $\sqrt{1-\Delta}$ makes sense as a pseudo-differential operator

- translation operator: for $a \in \mathbb{R}^d$, let $T_a(k) = e^{-iak} \Rightarrow T_a \in C^\infty_{pol}$

  $\Rightarrow$ for $\psi \in \mathcal{S}$, we find $\big(T_a(-i\nabla)\psi\big)(x) = (2\pi)^{-\frac{d}{2}} \int e^{ikx} e^{-iak} \hat{\psi}(k)\, dk$

  $$= (2\pi)^{-\frac{d}{2}} \int e^{ik(x-a)} \hat{\psi}(k)\, dk$$

  $$= \psi(x-a)$$

- free propagator: $P_f(k) = e^{-i\frac{k^2}{2}t} \Rightarrow P_f \in C^\infty_{pol}$

  $\Rightarrow$ solution to free Schrödinger equation is $\psi(t,x) = \left(\mathcal{F}^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F} \psi_0\right)(x)$

  $$= \left(P_f(t,-i\nabla)\psi_0\right)(x)$$

  $\Rightarrow \psi(t) = e^{\frac{i}{2}\Delta t} \psi(0)$

- heat equation: $\partial_t f(t,x) = \frac{1}{2}\Delta_x f(t,x)$

  $\Rightarrow W(t,k) = e^{-\frac{k^2}{2}t} \in C^\infty_{pol}$ for $t \geq 0$

  $\Rightarrow$ for $f(0,\cdot) = f_0 \in \mathcal{S}$, $t > 0$, we have $f(t) = e^{\frac{1}{2}\Delta t} f_0 = W(t,-i\nabla) f_0$

Definition 2.22:

The convolution of $f \in \mathcal{S}$ and $g \in \mathcal{S}$ is $(f * g)(x) := \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy$.